Unified Grounding

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Abstract: This paper offers a unification and systematization of the grounding approaches to truth, denotation, classes and abstraction. Its main innovation is a method for "kleenifying" bivalent semantics so as to ensure that the trivalent semantics used for various linguistic elements are perfectly analogous to the semantics used by Kripke, rather than relying on intuition to achieve similarity. The focus is on generalizing strong Kleene semantics, but one section is devoted to supervaluation, and the unification method also extends to weak Kleene semantics.

Kripke (1975) formulated a theory of truth and proposed a solution to the Liar Paradox and the other paradoxes that have to do with truth and non-wellfoundedness. In doing so, he made use of Kleene's (1952, §64) trivalent semantics for the connectives and their generalization to the quantifiers. The approach of Kripke and Kleene can be generalized by adding other syntactic elements to the formal language and interpreting them, like the truth predicate, with a partial evaluation that grows monotonically in a recursive "process", eventually yielding a fixed point. This has been done for denotation and the paradoxes thereof by Kremer (1990), Kroon (1991), and myself (Hansen 2012); for classes and Russell's Paradox by Maddy (1983, 2000); and for abstracta, by Horsten and Linnebo (2015) and other scholars cited in that paper.

This paper offers a unification of those four theories of grounding. The above-cited authors who take inspiration from Kripke all proceed by simply stipulating some trivalent semantics and claiming that the semantics for the additional linguistic resources, and the solutions to the paradoxes, are analogous to Kleene's and Kripke's. Instead of just stipulating some semantics that *seems* to accord with Kleene's, this paper systematizes by enforcing similarity in a canonical way: providing a precise definition of what it is to "kleenify" a given piece of classical semantics, and then extracting all the trivalent semantics needed to handle the paradoxes from "kleenifications" of bivalent semantics, thus ensuring that the solutions are uniform.

I will treat denotation in detail, as it has not yet – despite three attempts – been handled in a manner that is both fully satisfactory (in the sense of being made completely analogous to Kripke's theory[1]) and comprehensive (i.e. able to deal with all of the paradoxes of denotation). In contrast,

[1]Kroon's semantic clauses (1991, 28–32) are inconsistent; see in particular clauses (7) and (8). What seems to me to be the most natural way to regain consistency is to change "$\in$" in (2) to "$\notin$" and require that $d_1, \ldots, d_n \in D \cup \{\star\}$ in (8). In that case, predicates, including an x-refers-to-y predicate, have semantics that are analogous to weak Kleene,

Maddy and Horsten and Linnebo have formulated theories that are perfectly similar to Kripke's, and therefore there is less work left for this paper to do when it comes to classes and abstraction. However, they achieve this similarity merely by relying on intuitions about what semantics is analogous to Kripke's. Enforcing similarity in a systematic way is what is added here.

This paper is structured as follows. Section 1 is concerned with reformulating Kripke's theory in such a way that it (a) is recovered through kleenification and (b) is, in a certain sense, modular and thereby open to piecewise extension with linguistic resources beyond the truth predicate. Section 2 provides a theorem about fixed points and consistency that is general enough to cover both the theory presented in section 1 and the extensions that follow. Then, in section 3, the linguistic resources needed to formulate most of the paradoxes of denotation are added. One paradox of denotation is best formalized using classes and is therefore postponed to section 4, which is concerned with abstraction, and class abstraction in particular. Finally, in section 5, after having focused on strong Kleene for most of the paper, it is explained how the concept of kleenification extends to the supervaluation versions of Kripke's theory.

1 Kripke's theory reformulated

We define an *object language*, $\mathscr{L}$, that is similar to natural language in all aspects relevant to the paradoxes. One aspect of natural language relevant to Berry's Paradox is that not all natural numbers are describable with just one syllable; so in $\mathscr{L}$, not all natural numbers should be denoted by a constant. Similar points apply to König's Paradox and Richard's Paradox. On the other hand, having a constant for each object in the domain makes it possible to treat quantification substitutionally, which turns out to be significantly simpler than making use of assignments and the notion of satisfaction. We therefore also define an *extended language*, $\mathscr{L}^+$, that has that useful feature. The sets of *constants* for the two languages are denoted "$\mathcal{C}$" and "$\mathcal{C}^+$", respectively, and $\mathcal{C} \subset \mathcal{C}^+$. This is the only difference between the languages, so all other stipulations apply to both of them.

Let us first stipulate that for each $n \in \mathbb{N}$ there is a set $\mathcal{P}_n$ of *ordinary n-ary predicates*, supplemented with a unary, extraordinary predicate, T, the *truth predicate*. Reflecting the fact that there are only finitely many primitive proper names and predicates in natural language, $\mathcal{C}$ and $\bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ are assumed to be finite. We also have a countable set of *variables*. The syntaxes of $\mathscr{L}$ and of $\mathscr{L}^+$ are specified through definitions of *term* and

while the definite-description operator follows the connectives and quantifiers in being weak, strong or in-between.

Kremer (1990) is consistent but also relies on stipulations. Further, he stipulates a semantics for the definite-description operator which is a kleenification (see below for a definition) of a function that itself has "undefined" in its co-domain, leaving it unclear how that function would fit into a classical language.

In (Hansen 2012) I too just made stipulations, and my semantics for ordinary predicates was not a kleenification of anything, as I failed to take account of the possibility that a predicate may be indifferent about one or more of its terms, given the other terms.

wff (and later *class term*) as the smallest sets that satisfy the S-clauses to follow. These seven clauses give us what is needed for this section; more will be added later.

(Sc) Every constant is a term.

(Sx) Every variable is a term.

(S$\wedge$) If ϕ and ψ are wff's, then $\phi \wedge \psi$ is a wff.

(S$\neg$) If ϕ is a wff, then $\neg\phi$ is a wff.

(S$\forall$) If ϕ is a wff and x is a variable, then $\forall x\phi$ is a wff.

(SP) If P is an ordinary n-ary predicate and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is a wff.

(ST) If t is a term, then $T(t)$ is a wff.

Disjunction, the conditional, and the existential quantifier are defined in the usual way. When ϕ is a wff, x a variable, and c a constant, $\phi(x/c)$ is the wff that is identical to ϕ, with the (possible) exception that all free occurrences of x are replaced with c. A wff is a *sentence*, and a term is *closed*, if it does not contain any free variables. So far, a term's lack of free variables is equivalent to its not containing any variables at all, but that will change. Let $\mathcal{S}$ and $\mathcal{T}$ be the set of sentences and the set of closed terms, respectively, of $\mathscr{L}$, and likewise for $\mathcal{S}^+$ and $\mathcal{T}^+$. As $\mathcal{S} \subset \mathcal{S}^+$ and $\mathcal{T} \subset \mathcal{T}^+$, specifying the semantics for $\mathscr{L}^+$ also gives us the semantics for $\mathscr{L}$.

In our meta-language, the symbols $\top$, $\bot$ and $\star$ will be used for truth, falsehood, and undefinedness, respectively. For any set A, $A^\star$ will denote $A \cup \{\star\}$. When it is not explicitly stated that a given set contains $\star$, it does not.

A *model* is a pair $\mathfrak{M} = (D, I)$. D, the *domain*, is a set that includes as subsets ω_1 (for the purpose of Berry's and König's paradoxes), $\mathbb{R}$ (for the purpose of Richard's Paradox), and $\mathcal{S}^+ \cup \mathcal{T}^+ \cup \{\top, \bot\}$ (for reasons to be explained in connection with the definition of "evaluation", below). The *interpretation function*, I, satisfies the conditions given in the I-clauses to follow. Here is the first:

(Ic) The restriction of I to $\mathcal{C}^+$ is a bijection from $\mathcal{C}^+$ to D.

The surjectivity of the restriction of I to $\mathcal{C}^+$ is what allows us to treat quantification substitutionally. Nothing substantial hangs on the further stipulation of injectivity, but it streamlines the presentation, because (among other reasons) it allows us to use, for each $d \in D$, the symbol "c_d" for *the* constant such that $I(c_d) = d$. We let I interpret everything classically, hence this clause governing ordinary predicates:

(IP) For every $P \in \mathcal{P}_n$, $I(P)$ is a function from D^n to $\{\top, \bot\}$.

An *evaluation* is defined as a function from $D^\star$ to itself. The purpose of an evaluation is to assign truth values to sentences, and denotations to closed terms. Being a function from $D^\star$ to itself, it can do the former because the sentences are included in the domain of the function, and the truth values in the co-domain; and it can do the latter because the closed terms are also included in the domain, and every object they can potentially denote in the co-domain. We let the function's domain be all of $D^\star$ instead of just $\mathcal{S}^+ \cup \mathcal{T}^+$ because of cases such as a sentence of the form $T(r(t))$. Here, r is the reference function, to be formally introduced below, that maps an object to what it denotes. (The object to be mapped is the denotation of t, so $r(t)$ returns the denotation of the denotation of t, not the denotation of t.) Further, t is some complex term, formed by syntactic rules also not yet introduced, that is not given a denotation directly in the ground model, but which may or may not get one at some point in the recursion. If it ends up referring to something non-linguistic that does not itself have a reference, for example the Moon, then the semantics of $T(r(t))$ must still be well-defined. As this semantics is determined by what $r(t)$ refers to, the reference of $r(t)$ must be well-defined in all cases – or at least, ensuring that it is in a formal sense is the most elegant way of handling this kind of complication. Ergo, we let the Moon formally denote something, a dummy object. Falsehood, $\bot$, is reused in this role.[2] Similarly, while t is undefined, the value of undefined also needs formally to denote something, namely itself. In general, we want "everything" to denote "something", and that is why we settle on this definition of "evaluation".

The E-clauses that follow specify (relative to the model), recursively, a function, $\mathcal{J}$, called the *jump* that maps an evaluation $\mathcal{E}_-$ to another evaluation $\mathcal{E}$. The clauses covering the base cases of the recursive definition are listed here, with the rest following further down:

(Ec) $\mathcal{E}(c) = I(c)$ for all $c \in \mathcal{C}^+$

(Ed) $\mathcal{E}(d) = \bot$ for every element d of $D \setminus (\mathcal{S}^+ \cup \mathcal{T}^+)$

(E$\star$) $\mathcal{E}(\star) = \star$

For the remaining E-clauses we shall require the kleenified functions mentioned in the introduction, so it is time to define the concept of kleenification, along with a few associated concepts.

Definition 1. For any indexed family of sets $(A_j)_{j\in J}$ and any element $a = (a_j)_{j\in J}$ of $\prod_{j\in J} A_j^\star$, an element $\bar{a} = (\bar{a}_j)_{j\in J}$ of $\prod_{j\in J} A_j^\star$ is a *precisification* if, for all $j \in J$, if $a_j \neq \star$ then $a_j = \bar{a}_j$; and it is a *total precisification* if, additionally, $\bar{a}_j \in A_j$ for all $j \in J$.[3]

[2] Cf. Kremer (1990, 40), this means that the intuitive meaning of "there exists something that t refers to" is not captured by $\exists x(r(t) = x)$, but must be formalized $\exists x(r(t) = x \wedge x \neq c_\bot)$. Cf. Kroon (1991, 29), "$\bot$" can in this context be read as "determinately non-denoting", while "$\star$" means "undefined".

[3] We will not distinguish between the indexed family $\prod_{\{1,\dots,n\}} A$ and the Cartesian product A^n.

Kleene (1952, 334) defines a truth table as "regular" if "A given column (row) contains [$\top$] in the [$\star$] row (column), only if the column (row) consists entirely of [$\top$]'s; and likewise for [$\bot$]." This is the natural generalization:

Definition 2. A function $kf : \mathcal{A}_\star \to B^\star$, where $\mathcal{A}_\star$ is a subset of $\prod_{j \in J} A_j^\star$,[4] is *regular* if, for all elements $a = (a_j)_{j \in J}$ of $\mathcal{A}_\star$, $kf(a) = b \neq \star$ implies that for all precisifications $\bar{a}$ of a it is the case that $kf(\bar{a}) = b$.

What are now known as the "strong Kleene" ("weak Kleene") truth tables are the strongest (weakest) regular extensions of the classical truth tables in the sense that the function value is true or false (undefined) whenever possible. Thus, what is asserted below about regular functions applies to strong Kleene, weak Kleene and "everything in between".

The next definition is the central one, generalizing the procedure for obtaining the strong Kleene truth tables for the connectives from their classical counterparts.

Definition 3. For any given function $f : \mathcal{A} \to B$ or $f : \mathcal{A} \to B^\star$,[5] where $\mathcal{A}$ is a subset of $\prod_{j \in J} A_j$, the (*strongly*[6]) *kleenified-f* or the (*strong*) *kleenification of f* is defined to be the function $kf : \mathcal{A}_\star \to B^\star$, such that (1) $\mathcal{A}_\star$ is the set of those elements in $\prod_{j \in J} A_j^\star$ that have at least one precisification in $\mathcal{A}$ and (2) for all elements $a = (a_j)_{j \in J}$ of $\mathcal{A}_\star$, if there is a $b \in B$ such that for all total precisifications $\bar{a}$ of a it is the case that $f(\bar{a}) = b$, then $kf(a) = b$; otherwise $kf(a) = \star$.[7]

Obviously, any kleenified function is regular.

We immediately apply this. Instead of stipulating the trivalent semantics for conjunction directly, we only stipulate that the classical truth-function for it is $f_\wedge : \{\top, \bot\}^2 \to \{\top, \bot\}$ defined by $f_\wedge(\top, \top) = \top$ and $f_\wedge(\top, \bot) = f_\wedge(\bot, \top) = f_\wedge(\bot, \bot) = \bot$. We then apply the definition and obtain this semantic clause:

(KF$\wedge$) The kleenified conjunction-truth-function is $kf_\wedge : \{\top, \bot, \star\}^2 \to \{\top, \bot, \star\}$ given by

[4]In almost all applications we will make of this and the following definition, $\mathcal{A}_\star$ just is $\prod_{j \in J} A_j^\star$. The only examples where $\mathcal{A}_\star$ is a proper subset of $\prod_{j \in J} A_j^\star$ will come in the final paragraph of this paper.

[5]The latter option will only become relevant in connection with footnote 10.

[6]Similarly, the weakly kleenified-f can be defined as the function $wkf : \mathcal{A}_\star \to B^\star$ such that for all elements $a = (a_j)_{j \in J}$ of $\mathcal{A}_\star$, $wkf(a) = f(a)$, if $a \in \mathcal{A}$ and $wkf(a) = \star$ otherwise. However, we will not make use of this notion, so the word "strongly" will be suppressed in the following pages.

[7]Definitions 1–3 can alternatively be made in terms of information orderings. Let $A^\star$ be ordered such that $a \leq a'$ iff $a = \star$ or $a = a'$. Further, let $\prod_{j \in J} A_j^\star$ be ordered such that $(a_j)_{j \in J} \leq (a'_j)_{j \in J}$ iff $a_j \leq a'_j$ for all $j \in J$. In that case $(a'_j)_{j \in J}$ is a precisification of $(a_j)_{j \in J}$. If, further, $(a'_j)_{j \in J}$ is maximal, it is a total precisification of $(a_j)_{j \in J}$. A function is regular iff it is a homomorphism. Let the functions $\mathcal{A}_\star \to B^\star$, where $\mathcal{A}_\star$ is a subset of $\prod_{j \in J} A_j^\star$, be ordered such that $kf \leq kf'$ iff $kf(a) \leq kf'(a)$ for all $a \in \mathcal{A}_\star$. Given a function $f : \mathcal{A} \to B$, the strongly kleenified-f is the maximum among the regular functions of type $\mathcal{A}_\star \to B^\star$, for which it holds that the restriction of that function to $\mathcal{A}$ is identical to f. I leave the proof of equivalence to the reader.

- $kf_\wedge(\top, \top) = \top$,
- $kf_\wedge(\top, \bot) = kf_\wedge(\bot, \top) = kf_\wedge(\bot, \bot) = kf_\wedge(\bot, \star) = kf_\wedge(\star, \bot) = \bot$, and
- $kf_\wedge(\top, \star) = kf_\wedge(\star, \top) = kf_\wedge(\star, \star) = \star$.

Note that, unlike the S-, I-, and E-clauses, this and all the following KF-clauses are not definitions but theorems. (However, in all cases the proofs are so trivial that they will be omitted.)

Next, we substitute the kleenified function for the classical function to obtain the E-clause. The recursive clause for the classical evaluation of a conjunction is $\mathscr{E}(\phi \wedge \psi) = f_\wedge(\mathscr{E}(\phi), \mathscr{E}(\psi))$. Accordingly, the recursive clause for the Kripke-evaluation of a conjunction is (using implicit universal quantification over all sentences or terms of the given form from now on)

(E$\wedge$) $\mathcal{E}(\phi \wedge \psi) = kf_\wedge(\mathcal{E}(\phi), \mathcal{E}(\psi))$

We need to do the same for negation, the quantifier, the ordinary predicates, and the truth predicate. I will skip the first of these as it is trivial.

The (classical) universal-quantifier-truth-function is $f_\forall : \prod_D\{\top, \bot\} \to \{\top, \bot\}$ defined by $f_\forall((v_d)_{d\in D}) = \top$ if $v_d = \top$ for all $d \in D$ and $f_\forall((v_d)_{d\in D}) = \bot$ if $v_d = \bot$ for some $d \in D$, which is applied thus: $\mathscr{E}(\forall x\phi) = f_\forall((\mathscr{E}(\phi(x/c_d)))_{d\in D})$.

(KF$\forall$) The kleenified universal-quantifier-truth-function is $kf_\forall : \prod_D\{\top, \bot, \star\} \to \{\top, \bot, \star\}$ given, for all $(v_d)_{d\in D} \in \prod_D\{\top, \bot, \star\}$, by

- $kf_\forall((v_d)_{d\in D}) = \top$ if $v_d = \top$ for all $d \in D$,
- $kf_\forall((v_d)_{d\in D}) = \bot$ if $v_d = \bot$ for some $d \in D$, and
- $kf_\forall((v_d)_{d\in D}) = \star$ if $v_d \in \{\top, \star\}$ for all $d \in D$ and $v_d = \star$ for some $d \in D$.

(E$\forall$) $\mathcal{E}(\forall x\phi) = kf_\forall((\mathcal{E}(\phi(x/c_d)))_{d\in D})$

The (classical) truth-function for an arbitrary predicate P is $I(P) : D^n \to \{\top, \bot\}$, which is applied thus: $\mathscr{E}(P(t_1, \ldots, t_n)) = I(P)(\mathscr{E}(t_1), \ldots, \mathscr{E}(t_n))$.

(KFP) The kleenified-$I(P)$ is the function $kI(P) : (D^\star)^n \to \{\top, \bot, \star\}$ given, for all $(d_1, \ldots, d_n) \in (D^\star)^n$, by

- $kI(P)(d_1, \ldots, d_n) = \top$ if every total precisification $(\bar{d}_1, \ldots, \bar{d}_n)$ of $(d_1, \ldots, d_n)$ is such that $I(P)(\bar{d}_1, \ldots, \bar{d}_n) = \top$,
- $kI(P)(d_1, \ldots, d_n) = \bot$ if every total precisification $(\bar{d}_1, \ldots, \bar{d}_n)$ of $(d_1, \ldots, d_n)$ is such that $I(P)(\bar{d}_1, \ldots, \bar{d}_n) = \bot$, and
- $kI(P)(d_1, \ldots, d_n) = \star$ otherwise.

(EP) $\mathcal{E}(P(t_1, \ldots, t_n)) = kI(P)(\mathcal{E}(t_1), \ldots, \mathcal{E}(t_n))$.

The (classical) truth-predicate-truth-function is $f_T : D \to \{\top, \bot\}$ defined by $f_T(\top) = \top$ and $f_T(d) = \bot$ for all $d \in D \setminus \{\top\}$.

(KFT) The kleenified truth-predicate-truth-function is $kf_T : D^\star \to \{\top, \bot, \star\}$ given by

- $kf_T(\top) = \top$,
- $kf_T(d) = \bot$ for all $d \in D \setminus \{\top\}$, and
- $kf_T(\star) = \star$.

The recursive clause for the classical (Tarskian) evaluation of a truth sentence is $\mathscr{E}(T(t)) = f_T(\mathscr{E}_-(\mathscr{E}(t)))$. For Tarski, $\mathscr{E}(t)$ cannot be a sentence in the same language, but must be a sentence in a lower-level language (if it is a sentence at all); and $\mathscr{E}_-$ is then the evaluation function for that language. The recursive clause for the Kripke-evaluation of a truth sentence is

(ET) $\mathcal{E}(T(t)) = kf_T(\mathcal{E}_-(\mathcal{E}(t)))$.

Here $\mathcal{E}(t)$ is a sentence in the same language and $\mathcal{E}_-$ is the evaluation at the previous level.

To complete the reconstruction of Kripke's theory, we just need to define the hierarchy.

Definition 4. For every ordinal α we define the *evaluation with respect to the model* $\mathfrak{M}$ *and the level* α, $\mathcal{E}^\alpha$, by recursion: $\mathcal{E}^0$ is the "empty evaluation", i.e. the constant function mapping every element of $D^\star$ to $\star$. When α is a successor ordinal, $\mathcal{E}^\alpha = \mathcal{J}(\mathcal{E}^{\alpha-1})$. When α is a limit ordinal, $\mathcal{E}^\alpha$ is the "union" of all $\mathcal{E}^\eta$ for $\eta < \alpha$, in the sense that for all $d_1 \in D^\star$ and $d_2 \in D$, $\mathcal{E}^\alpha(d_1) = d_2$ iff there exists an $\eta < \alpha$ such that $\mathcal{E}^\eta(d_1) = d_2$, and $\mathcal{E}^\alpha(d_1) = \star$ iff there is no such d_2.[8]

As we will prove, the hierarchy ends in a fixed point, and we use the notation $[\![\phi]\!]$ for the value of ϕ in that fixed point. It also remains to be proved that this definition is legitimate, i.e. that there cannot be two different $d_2, d_2' \in D$ such that both $\mathcal{E}^\alpha(d_1) = d_2$ and $\mathcal{E}^\alpha(d_1) = d_2'$ should be the case according to the definition.

As mentioned in the introduction, the two languages are "modular": they can be extended with extra linguistic resources by adding an S-clause, an E-clause and a classical function to be kleenified plus, if presupposed by that function, an I-clause. As a result, the denotations of "$\mathscr{L}$" and "$\mathscr{L}^+$" as well as "sentence", "term", "$\mathcal{E}^\alpha$", etc. are relative to which modules are added. We could be more formally precise and avoid this relativity, but there is no need: the general points that will be made about the languages are independent of which addition modules are included.

[8]With reference to footnote 7, $\mathcal{E}^\alpha$, for α a limit ordinal, can simply be defined as the least upper bound of $\{\mathcal{E}^\eta | \eta < \alpha\}$.

2 Abstract fixed point theorem

We prove that the recursion is well-defined and ends in a consistent fixed point using a theorem that applies both to the interpreted language as it is specified so far, and to the extensions thereof as they are developed in sections 3 and 4. In the interest of generality, the theorem not only covers the case of the initial evaluation being empty, but any case where it is "sound". One more definition is needed: given a strictly partially ordered set $(A, <)$, let $A^{<a}$ be $\{a' \in A | a' < a\}$.

Theorem 1. *Let the following be given: (1) a well-founded, strictly partially ordered set $(D^\star, <)$, (2) for each $d \in D^\star$, a regular function $e_d : \prod_{D^\star} D^\star \times \prod_{D^{<d}} D^\star \to D^\star$, and 3) a function $\mathcal{E}^0 : D^\star \to D^\star$. Define, by recursion on $<$, the function $\mathcal{E}^1 : D^\star \to D^\star$ by $\mathcal{E}^1(d_0) = e_{d_0}((\mathcal{E}^0(d))_{d \in D^\star}, (\mathcal{E}^1(d))_{d \in D^{<d_0}})$. Assume that for each $d_0 \in D^\star$, if $\mathcal{E}^0(d_0) \neq \star$ then $\mathcal{E}^1(d_0) = \mathcal{E}^0(d_0)$. Then there is, for each ordinal $\alpha > 1$, a unique function $\mathcal{E}^\alpha : D^\star \to D^\star$ such that for all $d_0 \in D^\star$*

- *for successor α: $\mathcal{E}^\alpha(d_0) = e_{d_0}((\mathcal{E}^{\alpha-1}(d))_{d \in D^\star}, (\mathcal{E}^\alpha(d))_{d \in D^{<d_0}})$*
- *for limit α: $\mathcal{E}^\alpha(d_0) = \mathcal{E}^\beta(d_0)$ for all $\beta < \alpha$ for which $\mathcal{E}^\beta(d_0) \neq \star$, and if there is no such β then $\mathcal{E}^\alpha(d_0) = \star$*

Furthermore, there is an α such that for all $\beta > \alpha$ and for all $d \in D^\star$, $\mathcal{E}^\beta(d) = \mathcal{E}^\alpha(d)$.

Because this theorem is quite abstract, we need to account for how it applies. We do that before proving it. The theorem allows an element of the domain to be evaluated only by a function of the type $\prod_{D^\star} D^\star \times \prod_{D^{<d}} D^\star \to D^\star$, but none of the E-clauses do that. However, there is a canonical way to transform the functions referred to by the E-clauses (namely those in the KF-clauses) into functions of that type. First, we need to define the ordering $<$. Let it be the smallest transitive relation on $D^\star$ that satisfies the following: $\phi < \phi \wedge \psi$, $\psi < \phi \wedge \psi$, $\phi < \neg\phi$, $\phi(x/c) < \forall x\phi$, $t_1 < P(t_1, \ldots, t_n), \ldots, t_n < P(t_1, \ldots, t_n)$ and $t < T(t)$. As the relevant linguistic resources are introduced later in this paper, it must also satisfy these conditions: $t < r(t)$ (subsection 3.1), $\phi(x/c) < \imath x(\phi)$ (subsection 3.2), $ct < ct = ct'$ and $ct' < ct = ct'$ (section 4), $\phi(x/t) < t \in \hat{x}\phi$ (subsection 4.1) and $\phi(x/c) < cf(\hat{x}\phi)$ (subsection 4.2).

With that ordering in place, we can make the transformations. Let us take conjunction and the truth predicate as examples. To evaluate a conjunction, $\phi \wedge \psi$, we have a function $kf_\wedge$ that takes the values of ϕ and ψ as input, i.e., has $\{\top, \bot, \star\}^2$ as its domain. We can extend the domain to $(D^\star)^2$ by stipulating that the function value is $\bot$ if one or both of the arguments is not in $\{\top, \bot, \star\}$. Thus we have a regular function $kf'_\wedge : (D^\star)^2 \to D^\star$, which we can further transform into a function $e_{\phi\wedge\psi} : \prod_{D^\star} D^\star \times \prod_{D^{<d}} D^\star \to D^\star$ by "ignoring" most of the arguments: for all $\mathcal{E}_- \in \prod_{D^\star} D^\star$ and $\mathcal{E}|_{D^{<d}} \in \prod_{D^{<d}} D^\star$, $e_{\phi\wedge\psi}(\mathcal{E}_-, \mathcal{E}|_{D^{<d}}) = kf'_\wedge(\mathcal{E}|_{D^{<d}}(\phi), \mathcal{E}|_{D^{<d}}(\psi))$. This is again a regular function.

The evaluation of a sentence of the form $T(t)$ can also be put in the form of a function of the type $e_{T(t)} : \prod_{D^\star} D^\star \times \prod_{D^{<d}} D^\star \to D^\star$. Once again, let

$\mathcal{E}_- \in \prod_{D^\star} D^\star$ and $\mathcal{E}|_{D^{<d}} \in \prod_{D^{<d}} D^\star$. The closed term t is in the domain and is smaller than $T(t)$ by the ordering $<$. Hence, $\mathcal{E}|_{D^{<d}}(t)$ is well-defined and is an element of $D^\star$, and therefore $\mathcal{E}_-(\mathcal{E}|_{D^{<d}}(t))$ is well-defined. So define $e_{T(t)}(\mathcal{E}_-, \mathcal{E}|_{D^{<d}}) = \top$ if $\mathcal{E}_-(\mathcal{E}|_{D^{<d}}(t)) = \top$, $e_{T(t)}(\mathcal{E}_-, \mathcal{E}|_{D^{<d}}) = \star$ if $\mathcal{E}_-(\mathcal{E}|_{D^{<d}}(t)) = \star$, and $e_{T(t)}(\mathcal{E}_-, \mathcal{E}|_{D^{<d}}) = \bot$ if $\mathcal{E}_-(\mathcal{E}|_{D^{<d}}(t)) \notin \{\top, \star\}$. This is obviously equivalent with the rule for evaluation of $T(t)$ as specified by KFT and ET. And from these two examples for conjunction and the truth predicate, it should be clear how the rules for evaluation for the rest of the expressions of our language can be cast in the form of the theorem. The theorem is proved as follows:

Proof. The proof of the unique existence of the $\mathcal{E}^\alpha$s is by outer induction on α and inner induction on $d_0 \in D^\star$ as ordered by $<$. The induction proposition is a conjunction. The first conjunct is that there is a unique value of $\mathcal{E}^\alpha(d_0)$ that satisfies the conditions. The second conjunct is that there is an ordinal $\gamma \leq \alpha$ and a $b \in D^\star$ such that for all $\delta < \gamma$, $\mathcal{E}^\delta(d_0) = \star$, and for all δ such that $\gamma \leq \delta \leq \alpha$, $\mathcal{E}^\delta(d_0) = b$. The base case is trivial.

For the successor case, the first conjunct follows from the fact that at the given point in the recursion, the right-hand side of the equation in the second bullet point is well-defined. With regard to the second conjunct, we can distinguish two sub-cases, namely (1) that $\mathcal{E}^\delta(d_0) = \star$ for all $\delta < \alpha$, and (2) that there is an ordinal $\gamma \leq \alpha - 1$ and a $b \in D$ (note: not "$b \in D^\star$") such that for all $\delta < \gamma$, $\mathcal{E}^\delta(d_0) = \star$, and for all δ such that $\gamma \leq \delta \leq \alpha$, $\mathcal{E}^\delta(d_0) = b$. In the first sub-case, any value of $\mathcal{E}^\alpha(d_0)$ will make the conjunct true. In the second sub-case, assume first that γ is a successor ordinal, so that $\mathcal{E}^\gamma(d_0) = e_{d_0}((\mathcal{E}^{\gamma-1}(d))_{d\in D^\star}, (\mathcal{E}^\gamma(d))_{d\in D^{<d_0}})$. Then the set of precisifications of $((\mathcal{E}^{\alpha-1}(d))_{d\in D^\star}, (\mathcal{E}^\alpha(d))_{d\in D^{<d_0}})$ is, by the induction hypothesis, a subset of the set of precisifications of $((\mathcal{E}^{\gamma-1}(d))_{d\in D^\star}, (\mathcal{E}^\gamma(d))_{d\in D^{<d_0}})$. Ergo, as e_{d_0} is a regular function, $e_{d_0}((\mathcal{E}^{\alpha-1}(d))_{d\in D^\star}, (\mathcal{E}^\alpha(d))_{d\in D^{<d_0}})$ is equal to $e_{d_0}((\mathcal{E}^{\gamma-1}(d))_{d\in D^\star}, (\mathcal{E}^\gamma(d))_{d\in D^{<d_0}})$; so the second conjunct is true. If γ is instead equal to 0, then run the same argument but with 1 in place of γ. It is obvious that γ cannot be a limit ordinal different from 0.

The limit case follows directly from the induction hypothesis.

As D is set-sized, the existence of an α such that for all $\beta > \alpha$ and for all $d \in D^\star$, $\mathcal{E}^\beta(d) = \mathcal{E}^\alpha(d)$ follows by the usual cardinality argument. □

3 Denotation

We will now extend the language and apply it to treat the paradoxes.

3.1 Reference

The key to treating the paradoxes of denotation in the same way as the paradoxes of truth is to have a reference function in the language that works in a way that is similar to the truth predicate. The truth-predicate-truth-function is the identity function except for those values that it is not "supposed" to be applied to, i.e. non-truth-values. It is only because it is syntactically possible to apply the truth predicate to a term that refers to something that is not a sentence that the full function is not the identity function. The reference function works exactly like the truth predicate except for the exception:

(Sr) If t is a term, then $r(t)$ is a term.

The (classical) reference function is the identity function $f_r : D \to D$, which is applied thus: $\mathscr{E}(r(t)) = f_r(\mathscr{E}_-(\mathscr{E}(t)))$.

(KFr) The kleenified reference function is the identity function $kf_r : D^\star \to D^\star$.[9]

(Er) $\mathcal{E}(r(t)) = kf_r(\mathcal{E}_-(\mathcal{E}(t)))$.

Here is an example: Let t_0 be a constant that denotes the definite description (to be introduced into our formal language in the next subsection) "the smallest perfect number". Then $kf_r(\mathcal{E}_-(\mathcal{E}(t_0))) = kf_r(\mathcal{E}_-(\text{"the smallest perfect number"})) = kf_r(6) = 6$.

The Tarskian reason for using $\mathscr{E}_-$ instead of a second application of $\mathscr{E}$ is that the paradoxes of reference show that a classical and self-referential language cannot contain its own reference function, just as the Liar and the other paradoxes of truth show that it cannot contain its own truth predicate. The Kripkean reason for using $\mathcal{E}_-$ is that $\mathcal{E}(t)$ may be of higher syntactic complexity than $r(t)$; so for the recursion to be well-defined, we have to look back a level.

3.2 Definite descriptions

(S$\imath$) If ϕ is a wff and x is a variable, then $\imath x(\phi)$ is a term.

The (classical) definite-description-function is $f_\imath : \prod_D\{\top, \bot\} \to D$ defined, for all $(v_d)_{d\in D} \in \prod_D\{\top, \bot\}$, by $f_\imath((v_d)_{d\in D}) = d_0$ if $v_{d_0} = \top$ and $v_d = \bot$ for all $d \neq d_0$, and $f_\imath((v_d)_{d\in D}) = \bot$[10] if $v_d = \bot$ for all d or $v_d = \top$ for more than one d, which is applied thus: $\mathscr{E}(\imath x(\phi)) = f_\imath((\mathscr{E}(\phi(x/c_d)))_{d\in D})$.

(KF$\imath$) The kleenified definite-description-function is $kf_\imath : \prod_D\{\top, \bot, \star\} \to D^\star$ given, for all $(v_d)_{d\in D} \in \prod_D\{\top, \bot, \star\}$, by

- $kf_\imath((v_d)_{d\in D}) = d_0$ if $v_{d_0} = \top$ and $v_d = \bot$ for all $d \neq d_0$,

[9] This is because D contains more than one element. Had it contained only one, the kleenified function would map $\star$ to that element.

[10] Recall footnote 2: as a semantic value of a term, $\bot$ means "determinately non-denoting".
We could have followed Frege here and used $\star$ instead of $\bot$ so that a sentence such as "the largest prime is odd" would come out undefined. Note that that option is already covered by Definition 3, as it allows the function to be kleenified to have $\star$ in the co-domain.

- $kf_\imath((v_d)_{d\in D}) = \bot$ if $v_d = \bot$ for all d or $v_d = \top$ for more than one d, and
- $kf_\imath((v_d)_{d\in D}) = \star$ otherwise.

(E$\imath$) $\mathcal{E}(\imath x(\phi)) = kf_\imath((\mathcal{E}(\phi(x/c_d)))_{d\in D})$.

3.3 Berry's Paradox

With the reference function and the definite-description operator in place, it becomes possible to formalize the description responsible for Berry's Paradox.

Berry's Paradox: *The definite description*

> *Berry's description: the least integer not describable in fewer than twenty syllables*

is a description of nineteen syllables. So the least integer not describable in fewer than twenty syllables is describable in only nineteen syllables.[11]

Formalization: Let n, m, and x be variables, S and N be unary predicates, and $\equiv$ and $\geq$ be binary predicates (written in-fix), such that $I(S)$ is the set of "short" closed terms of $\mathscr{L}$, i.e. those that contain 67 primitive symbols or fewer; $I(N)$ is the set of natural numbers; $I(\geq)$ is the relation "larger than or equal to" on the set of natural numbers ($\bot$ whenever one of the relata is not a natural number); and $I(\equiv)$ is the identity relation (reserving the symbol "=" for later).

We can formalize "The natural number n is not denoted by a short closed term" thus:

$$N(n) \land \forall x(r(x) \equiv n \to \neg S(x))$$

Therefore, "n is the least natural number that is not denoted by a short closed term" can be formalized

$$(N(n) \land \forall x(r(x) \equiv n \to \neg S(x))) \land \\ \forall m(N(m) \land \forall x(r(x) \equiv m \to \neg S(x)) \to m \geq n).$$

Ergo, Berry's description in a version that substitutes length of formal expressions for number of syllables, "the least natural number that is not denoted by a short closed term", can be formalized as (B):

$$\imath n((N(n) \land \forall x(r(x) \equiv n \to \neg S(x))) \land \\ \forall m(N(m) \land \forall x(r(x) \equiv m \to \neg S(x)) \to m \geq n)) \quad \text{(B)}$$

(B) contains no constants and is therefore a closed term of $\mathscr{L}$ with 67 primitive symbols.

Proof of undefinedness: That $[\![(\mathrm{B})]\!] = \star$ is proved by induction on the levels. The base and limit cases are trivial. For the successor case, assume that $\mathcal{E}^{\alpha-1}((\mathrm{B})) = \star$. From this it follows that

$$\mathcal{E}^{\alpha}(r(c_{(\mathrm{B})})) = kf_r(\mathcal{E}^{\alpha-1}(\mathcal{E}^{\alpha}(c_{(\mathrm{B})}))) = kf_r(\mathcal{E}^{\alpha-1}((\mathrm{B}))) = kf_r(\star) = \star.$$

[11] Russell 1908.

Given that $\mathcal{E}^\alpha(\neg S(c_{(\mathrm{B})})) = \bot$, it can then be inferred that for all $c \in \mathcal{C}^+$, $\mathcal{E}^\alpha(r(c_{(\mathrm{B})}) \equiv c \to \neg S(c_{(\mathrm{B})})) = \star$, and thus that

$$\mathcal{E}^\alpha((N(n) \wedge \forall x(r(x) \equiv n \to \neg S(x))) \wedge \\ \forall m(N(m) \wedge \forall x(r(x) \equiv m \to \neg S(x)) \to m \geq n)) \neq \top.$$

Ergo, neither the first bullet point nor the second disjunct of the second bullet point of KF*7* is satisfied. We need to show that the first disjunct of the second bullet point also is not, because then the third bullet point is.

Closed terms that can be obtained from each other through the renaming of variables are obviously given the same value at each level, so consider them identical. Then there are only finitely many closed terms consisting of at most 67 primitive symbols in $\mathscr{L}$. Let n be a natural number that is not denoted by one of these at level $\alpha - 1$, and let $\bar{n}$ be a constant for n in $\mathscr{L}^+$. It follows that $\mathcal{E}^\alpha(N(\bar{n})) = \top$ and that $\mathcal{E}^\alpha(\forall x(r(x) \equiv \bar{n} \to \neg S(x))) \neq \bot$. Because of the rule for conjunction, we have now reduced the problem to showing that

$$\mathcal{E}^\alpha(\forall m(N(m) \wedge \forall x(r(x) \equiv m \to \neg S(x)) \to m \geq n)) \neq \bot.$$

However, that follows from the already established undefinedness of $r(c_{(\mathrm{B})}) \equiv c \to \neg S(c_{(\mathrm{B})})$ for all $c \in \mathcal{C}^+$. So it can be concluded that $\mathcal{E}^\alpha((\mathrm{B})) = \star$.

3.4 Other paradoxes

Having dealt with Berry's Paradox in detail, I believe I can safely leave a number of other paradoxes of denotation as exercises for the reader: König's Paradox[12], Hilbert and Bernays' Paradox[13], Uzquiano's Paradox[14] and Simmons' Paradox[15]. (Formalizing Hilbert and Bernays' Paradox and Simmons' Paradox requires the addition of function symbols to the language and kleenification of their semantics.) Richard's Paradox is treated at the end of the following section.

4 Abstraction

In order to formulate the definite description responsible for the last of the paradoxes of reference, Richard's, we must have class functions. We therefore postpone this paradox until we have treated the more fundamental aspects of classes. Classes are formed by abstraction and will be our primary example of abstracta.

To extend our language with some form of abstraction, we need an operator § that transforms a linguistic item a of a specified sort into an abstraction

[12] König 1905.

[13] Originally presented in (Bernays and Hilbert 1939, 263–278); natural language formulation in (Priest 2006).

[14] Uzquiano 2004.

[15] Simmons 2005.

term $\S a$, along with a truth function that results in an equivalence relation indicating which pairs of abstraction terms $\S a$ and $\S b$ make the sentence $\S a = \S b$ true. As long as this truth function takes as its inputs only the semantic values of the sentences and terms, the function can be kleenified. For instance, if the domain contains lines and there is a binary predicate P meaning "are parallel", then the abstraction operator d for "the direction of" can be introduced, together with a truth function to determine the truth value of $d(a) = d(b)$ – which is simply the identity function applied to the evaluation of the sentence $P(a, b)$, and the latter can of course be kleenified. Likewise, numbers could be introduced using Hume's Principle: the abstraction operator is #, which takes a wff with one free variable to a number term; and the truth function is one that serves to make $\#\phi(x) = \#\psi(y)$ true iff there is a bijection between the true sentences of the form $\phi(x/c_d)$ and the true sentences of the form $\psi(y/c_d)$, and such a truth function is also kleenifiable. We will not formally introduce either of these things into our language, as our focus is on paradoxes and generalizing Kripke's solution. We therefore only formally introduce classes:

(Sˆ) If ϕ is a wff and x is a variable, then $\hat{x}\phi$ is a class term.

(S=) If ct and ct' are class terms, then $ct = ct'$ is a wff.

The (classical) class-identity-truth-function is the function $f_= : \prod_D(\{\top, \bot\}^2) \to \{\top, \bot\}$ defined, for all $(v_d, v'_d)_{d\in D} \in \prod_D(\{\top, \bot\}^2)$, by $f_=((v_d, v'_d)_{d\in D}) = \top$ if $v_d = v'_d$ for all $d \in D$ and $f_=((v_d, v'_d)_{d\in D}) = \bot$ otherwise, which is applied thus: $\mathscr{E}(\hat{x}\phi = \hat{y}\psi) = f_=((\mathscr{E}(\phi(x/c_d)), \mathscr{E}(\psi(y/c_d)))_{d\in D})$.

(KF=) The kleenified class-identity-truth-function is $kf_= : \prod_D(\{\top, \bot, \star\}^2) \to \{\top, \bot, \star\}$ given, for all $(v_d, v'_d)_{d\in D} \in \prod_D(\{\top, \bot, \star\}^2)$, by

- $kf_=((v_d, v'_d)_{d\in D}) = \top$ if $v_d = v'_d = \top$ or $v_d = v'_d = \bot$ for all $d \in D$,
- $kf_=((v_d, v'_d)_{d\in D}) = \bot$ if ($v_d = \top$ and $v'_d = \bot$) or ($v_d = \bot$ and $v'_d = \top$) for some $d \in D$, and
- $kf_=((v_d, v'_d)_{d\in D}) = \star$ otherwise.

(E=) $\mathcal{E}(\hat{x}\phi = \hat{y}\psi) = kf_=((\mathcal{E}(\phi(x/c_d)), \mathcal{E}(\psi(y/c_d)))_{d\in D})$.

I refer the reader to (Horsten and Linnebo 2015) for more on identification of classes.

4.1 Class membership and Russell's Paradox

So far, classes in our language are practically inert, only being allowed to occur flanking the identity symbol. This is changed by adding class membership statements, following (Maddy 1983) and (Maddy 2000). This is the third element of our language for which the semantics has to be specified by reference to the previous level because its value may depend on the value of a sentence of higher syntactic complexity. And again the semantic function is an identity function (with no exceptions, like reference and unlike truth).

(S$\in$) If t is a term and ct is a class term, then $t \in ct$ is a wff.

The (classical) membership-truth-function is the identity function $f_\in : \{\top, \bot\} \to \{\top, \bot\}$, which is applied thus: $\mathscr{E}(t \in \hat{x}\phi) = f_\in(\mathscr{E}_-(\phi(x/t)))$.

(KF$\in$) The kleenified membership-truth-function is the identity function $kf_\in : \{\top, \bot, \star\} \to \{\top, \bot, \star\}$.

(E$\in$) $\mathcal{E}(t \in \hat{x}\phi) = kf_\in(\mathcal{E}_-(\phi(x/t)))$.

In order to allow classes to be members of classes, we further stipulate that

(S$\hat{2}$) Every class term is a term.

However, when classes as well as non-classes can be elements of classes, the semantics of $\phi(x/t)$ must be well-defined also when t is a class term, i.e. a class term must make sense in any syntactic position where a(n ordinary) term can occur. We achieve that with this simple stipulation, reflecting the idea that a class term does not denote any individual:

(E$\hat{}$) $\mathcal{E}(ct) = \bot$ for every class term ct.

The intuitive denotation of a class term is modeled indirectly with $(\mathcal{E}(\phi(x/c_d)))_{d \in D}$ instead of $\mathcal{E}(\hat{x}\phi)$. (And since $\mathcal{E}(c_{ct}) = ct \in \mathcal{D}$, predicates can still be used to express something about class terms.)

Russell's Class is $\hat{x}(x \notin x)$, so Russell's Paradox arises from this sentence: $\hat{x}(x \notin x) \in \hat{x}(x \notin x)$. It is of course undefined, as is proved in (Maddy 1983).

4.2 Class functions

Returning to paradoxes of reference, we need diagonalization to formulate Richard's Paradox, and in order to model diagonalization we need functions that apply to classes instead of to individual elements of the domain. Intuitively, a class function maps a class of objects from the domain to one object from the domain (we will only introduce unary class functions). Identifying such a class of objects with the element of $\prod_D\{\top, \bot\}$ for which the elements of the class index the $\top$-entries and the non-elements index the $\bot$-entries, we arrive at the following formal stipulations:

Let there be a finite set $\mathcal{CF}$ of *class function symbols*.

(SCF) If cf is a class function symbol and ct is a class term, then $cf(ct)$ is a term.

(ICF) For every $cf \in \mathcal{CF}$, $I(cf)$ is a function from $\prod_D\{\top, \bot\}$ to D.

A classical class-function is of the form $I(cf) : \prod_D\{\top, \bot\} \to D$, which is applied thus: $\mathscr{E}(cf(\hat{x}\phi)) = I(cf)((\mathscr{E}(\phi(x/c_d)))_{d \in D})$.

(KFCF) The kleenified-$I(cf)$ is the function $kI(cf) : \prod_D\{\top, \bot, \star\} \to D^\star$ given, for all $(v_d)_{d\in D} \in \prod_D\{\top, \bot, \star\}$, by

- $kI(cf)((v_d)_{d\in D}) = d_0$ if every total precisification $(\bar{v}_d)_{d\in D}$ of $(v_d)_{d\in D}$ is such that $I(cf)((\bar{v}_d)_{d\in D}) = d_0$, and
- $kI(cf)((v_d)_{d\in D}) = \star$ if there is no such d_0.

(ECF) $\mathcal{E}(cf(\hat{x}\phi)) = kI(cf)((\mathcal{E}(\phi(x/c_d)))_{d\in D})$.

4.3 Richard's Paradox

Richard's Paradox: *The set of all reals that are definable by a definite description in English is countable, so it can be enumerated. Fix a method of enumeration.*[16] *Given an enumeration of a set of reals, we can diagonalize out of it; fix also a specific method of diagonalization.*[17] *This means that the definite description*

> *Richard's description: the diagonalisation of the enumeration of the set of all definable reals,*

is both not definable (by the diagonalisation) and definable (by Richard's description).[18]

Formalization: In essence, diagonalization is a function that maps countable sets of real numbers to a real number not in that set. The procedure of changing the nth decimal of the nth real number in some numbering of them is one way to do that, but we can abstract from the specifics here. We can simply stipulate that d is a class-function symbol such that $I(d)$, informally characterized, is a function that sends each proper subset of $\mathbb{R}$ to a real number not in that subset. Formally, $I(d)$ is a function from $\prod_D\{\top, \bot\}$ to D such that each element of $\prod_D\{\top, \bot\}$ for which it holds that (1) all entries indexed by non-reals are $\bot$ and (2) at least one entry indexed by a real is $\bot$ too is mapped to a real that indexes an entry which is $\bot$. For definiteness, say that every other element of $\prod_D\{\top, \bot\}$ is mapped to $\bot$.

If we further let x and y be variables and C and P be unary predicates such that $I(C) = \mathcal{T}$ and $I(P) = \mathbb{R}$, then Richard's description can be formalized like this:

$$d(\hat{x}(P(x) \land \exists y(C(y) \land r(y) \equiv x))) \tag{R}$$

[16]For example the enumeration which results from ordering the reals primarily by the length of, and secondarily alphabetically by, their definite description (or the first of these by the same ordering when there are more definite descriptions of the same number); i.e. a number with a shorter description always comes before a number with (only) a longer description, and numbers with (shortest) descriptions of the same length are ordered alphabetically.

[17]For example, take the real number in the interval $[0, 1)$ which for its nth decimal has the sum of 2 and the nth decimal of the nth real in the enumeration under the convention that $2 + 8 = 0$ and $2 + 9 = 1$. This number is not in the set, since it differs from each number in the set on at least one decimal. (The decimal representation of a real is unique. There is only the one exception, made irrelevant by the use of 2 as the one addend, that a real with a decimal representation with a "tail" of 0's also has a decimal representation with a "tail" of 9's and *vice versa*.)

[18]Richard 1905.

Proof of undefinedness: First, note that the kleenified function $kI(d)$ is such that an element of $\prod_D \{\top, \bot, \star\}$ for which it holds that

1. every entry indexed by a non-real is $\bot$,
2. at most countably many entries indexed by reals are $\top$, and
3. the rest of the entries indexed by reals are $\star$

is mapped to $\star$. This follows from the fact that the different total precisifications of such an element are mapped to different real numbers (or, in one case, to $\bot$).

By induction on the levels, it can be shown that (R) is undefined. The base case and the limit case are trivial. At every successor level, the undefinedness of (R) follows from the fact that it is an element of $\prod_D \{\top, \bot, \star\}$ with the characteristics 1.-3. that $kI(d)$ is applied to. This can be seen from the following:

1. $P(c)$ and therefore $P(c) \wedge \exists y(C(y) \wedge r(y) \equiv c)$ are false for any constant c that denotes a non-real.
2. There are only countably many closed terms t such that $C(c_t)$ is true, and therefore at most countably many reals r such that $P(c_r) \wedge \exists y(C(y) \wedge r(y) \equiv c_r)$ is true.
3. $P(c) \wedge \exists y(C(y) \wedge r(y) \equiv c)$ is not false for any c denoting a real. For by the induction hypothesis, $r(c_{(\mathrm{R})})$ is undefined, so $r(c_{(\mathrm{R})}) \equiv c$ is undefined, so $C(c_{(\mathrm{R})}) \wedge r(c_{(\mathrm{R})}) \equiv c$ is undefined ((R) contains no constants and is an element of $\mathcal{T}$), so $\exists y(C(y) \wedge r(y) \equiv c)$ is not false. As $P(c)$ is true, $P(c) \wedge \exists y(C(y) \wedge r(y) \equiv c)$ is not false.

5 Kripke's supervaluation

What we have accomplished is a completely unified Kripkean approach to truth, denotation, classes and abstraction. The means to that end was the concept of kleenification. That concept can also be put into service to achieve unification along another dimension, namely between the strong Kleene scheme – which we have focused on until now – and the seemingly quite different supervaluation scheme. The latter can also be recovered by kleenification of bivalent semantics. Let me explain that informally and through examples first.

When a model is considered fixed, each sentence corresponds to a function that maps $\mathcal{E}_-$ to a truth value. A sentence that does not contain T, r or $\in$ corresponds to a constant function. For instance, $B(s)$, where B means "is black" and s means "snow", maps every possible $\mathcal{E}_-$ to $\bot$, assuming a natural model. On the other hand, $T(c_\phi)$ is associated – in the classical case – with a function that maps evaluations $\mathscr{E}_-$ according to which ϕ is true to $\top$, and evaluations $\mathscr{E}_-$ according to which ϕ is false to $\bot$. The function that corresponds to complex sentences is composed out of the functions associated with its constituents. For classical semantics, the method for composing is given by the classical functions and the classical evaluation clauses above. For instance, $\neg T(c_\phi)$ corresponds to the function that maps $\mathscr{E}_-$ to $\bot$ ($\top$) if ϕ is true (false) according to $\mathscr{E}_-$, which means that $T(c_\phi) \wedge \neg T(c_\phi)$ also corresponds to the constant function that maps every possible $\mathscr{E}_-$ to $\bot$.

When we kleenify the functions f_T, $f_\neg$, and $f_\wedge$ individually and then compose the results, as we have done above, the function for $T(c_\phi) \wedge \neg T(c_\phi)$ that comes out of it is not constant; it maps evaluations $\mathcal{E}_-$ according to which ϕ is undefined to $\star$. But if, instead, we first compose the classical functions and *then* kleenify, we get a different result. Kleenification of a constant function yields a constant function, so again $T(c_\phi) \wedge \neg T(c_\phi)$ is mapped to $\bot$ in all cases.

In section 1 we defined the jump, $\mathcal{J}$, as a function from evaluations to evaluations through the recursive definition of $\mathcal{E}$ given $\mathcal{E}_-$. Let the classical jump (think of the revision rule of Gupta's (1982) revision theory), $\mathscr{J}$, be defined similarly as the function that maps the classical evaluation (a funtion from D to D) $\mathscr{E}_-$ to the classical evaluation $\mathscr{E}$.[19] Then consider, for each $d \in D$, the function σ_d defined by $\sigma_d(\mathscr{E}_-) = \mathscr{J}(\mathscr{E}_-)(d)$. This function is of type $\prod_D D \to D$. The kleenification of σ_d, $k\sigma_d$, is therefore of type $\prod_D D^\star \to D^\star$. What does it take for $k\sigma_d(\mathcal{E}_-)$ to be an element of D rather than equal to $\star$? That happens when all the classical evaluations that are precisifications of $\mathcal{E}_-$ are mapped to the same element of D. That is, when d is a sentence, $\mathcal{E}_-$ is mapped to $\top$ ($\bot$) iff all the classical evaluations that are precisifications of $\mathcal{E}_-$ are mapped to $\top$ ($\bot$). That is precisely the simple kind of supervaluation discusses by Kripke.

This simple supervaluation theory employs quantification over *all* total precisifications of the given evaluation. Kripke considers two variations, in which the quantification is restricted to consistent and maximally consistent precisifications, respectively. To reconstruct those variations in the present approach we finally need to make use of the full generality of the definition of "kleenification". Recall that the definition allows for kleenification of not only functions of the type $\prod_{j\in J} A_j \to B$, but also of functions for which the domain is a proper subset of $\prod_{j\in J} A_j$. The effect of "removing" elements of $\prod_{j\in J} A_j$ is that fewer precisifications of a given element of $\prod_{j\in J} A_j^\star$ have to "agree" in order to map that element to a value in B. The precisifications to be disregarded could for example be those that are not consistent or those that are not maximally consistent. Therefore, the more advanced forms of supervaluation are achieved in the same way as the simple form, except that instead of kleenifying the functions σ_d, we kleenify the restriction of σ_d to the subsets of $\prod_D D$ consisting of (for the first variation) those functions that do not map any sentence together with its negation to $\top$ or (for the second variation) those functions that for each sentence maps that sentence to $\top$ and its negation to $\bot$ or vice versa.

[19] The clauses (Ec), (Ed) and (E$\hat{\ }$) holds also when "$\mathcal{E}$" is replaced with "$\mathscr{E}$". With that stipulation, $\mathscr{J}$ has been completely specified.

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