On Fair Countable Lotteries

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Abstract: Two reverse supertasks—one new and one invented by Pérez Laraudogoitia (2014)—are discussed. Contra Kerkvliet (2016) and Pérez Laraudogoitia, it is argued that these supertasks cannot be used to conduct fair infinite lotteries, i.e., lotteries on the set of natural numbers with a uniform probability distribution. The new supertask involves an infinity of gods who collectively select a natural number by each removing one ball from a collection of initially infinitely many balls in a reverse omega-sequence of actions.

Keywords: Reverse supertasks, uniform probability distributions, countable additivity axiom

1 A story

The Society of Gods—which last gathered its denumerably infinitely many card-holding members in 1964 to demonstrate how they could stop a walking man by the sheer force of their intentions[1]—has just reconvened. This time, they plan to hold a fair infinite lottery. One can enter the lottery by buying a ticket with any natural number printed on it. The gods therefore produce an infinity of balls, one for each of the natural numbers and each marked with that number, and use them to draw a winner.

The membership cards of the Society are also numbered by the naturals, and according to that numbering the collection of balls is passed from god to god in reverse order, such that in the span of one minute the urn has been in the possession of each member of the Society. God number 1 will be passed the urn at $t = 1$, and it is his job to remove all but one ball from the collection; the remaining ball contains the winning number. Before that, the urn will be with god number 2, who gets it at $t = \frac{1}{2}$ and reduces the number of balls to 2. The third god will get the collection at $t = \frac{1}{3}$ and he must leave 3 balls. In general, god number n is handed the collection at $t = \frac{1}{n}$, reduces the number of balls in it to n and, if $n \neq 1$, passes it on to

[1] The minutes of that meeting can be found on pages 259–260 of (Benardete 1964).

god number $n-1$. At $t=0$, the collection contains all the infinitely many balls.

However, the gods do not simply give themselves the instruction *when god n receives $n+1$ balls, he must leave n balls in the collection.* If they did, it would not ensure that each god n actually receives $n+1$ balls. They might, for example, all receive the collection still containing all the balls, thus rendering them unable to follow the instruction, because all the previous gods had received it still containing all the balls and been unable to follow the instruction. Or, more generally, a god might receive the wrong number of balls because all the previous gods had received the wrong number of balls. The gods guard themselves against this danger by instead giving themselves this more complex instruction: *if god n receives $n+1$ balls, he must leave n balls according to rule R* (to be specified)*; if he receives any other number (including zero), he passes on n balls numbered 1 through n, adding and removing balls as necessary.* If any god were to follow the consequent of the second part of the instruction because he receives the wrong number of balls, he would affect the contents of the collection in a way that might not be in accordance with rule R, thus ruining the experiment. However, the complex instruction prevents this from happening. The instruction ensures that each god leaves the right number of balls, no matter what. Ergo, each god will act on the first part of the instruction, and the second part is never actually used.[2] In this way, the gods ensure that the experiment runs as intended.

They also all agree to remove balls in a random way. That is, they agree that R should be "do it randomly". Or to be more precise, each god promises to employ a uniform probability distribution over the finite number of options he has for how to remove balls. That way, they believe, they ensure that, collectively, they conduct a fair infinite lottery on the natural numbers.

2 Finite additivity

According to de Finetti (1974), it ought to be considered rationally permissible to have a uniform probability distribution on the elements of a countably infinite outcome space: for instance, in the case of a lottery on the natural numbers. Such a distribution must assign probability 0 to each $n \in \mathbb{N}$ but probability 1 to $\mathbb{N}$, in violation of the countable additivity principle. Only the weaker principle of finite additivity holds, he believes.

De Finetti finds it unreasonable that—as is the case according to Kolmogorov's (1933) definition—any probability distribution on the natural numbers has to be "biased" towards the low numbers, in the sense that

[2] In this respect they take inspiration from (Pérez Laraudogoitia 2011).

for any real number $p \in [0, 1)$, there must be a natural number n such that the probability that the outcome of such a lottery is less than n is equal to or higher than p.

Two recent papers have offered support for de Finetti's claim, both of them through considerations concerned with reverse supertasks that seem to provide mechanisms for realizing such fair lotteries on the natural numbers. Pérez Laraudogoitia (2014) presents a supertask of his own and claims that he can use it to disprove the principle of countable additivity. Kerkvliet (2016) provides a mathematical model which violates countable additivity of the supertask described above and asserts that the model is "not controversial".

I will first summarize Kerkvliet's basic modeling assumption, along with his results. In section 3, I will show that there is a fatal flaw in Kerkvliet's model. Then, in section 4, I will examine the supertask devised by Pérez Laraudogoitia and his claim that it can be used to disprove countable additivity. The final section assesses the dialectical situation.

For present purposes, Kerkvliet's paper can be summarized quite briefly. His model is centered on a constraint that is a formal equivalent of the stipulation that each god makes a fair and random selection: conditional on any given god n receiving any given ball number k as part of the collection passed to him by god $n + 1$, the probability that ball number k remains in the collection when it is passed to god number $n-1$ (or in the case of god 1: is the final ball) must be $\frac{n}{n+1}$. Based on this constraint, Kerkvliet proves two things: first, that for any finite subset of $\mathbb{N}$, the probability of the winning number being in that subset is 0—as required by de Finetti. Second (and actually not relevant for present purposes, but it would be an odd paper summary that did not mention the primary point of the paper in question), he proves that the probability distribution is otherwise underdetermined, in the sense that for any subset of $\mathbb{N}$ which is neither finite nor co-finite, the probability of the winning number being in that subset can be any number in the interval $[0, 1]$.

3 Deterministic and stochastic versions of the scenario

The reader probably noticed that if the final paragraph is disregarded, section 1 contains a description of a generic scenario that is more general than a lottery. The "R" can be instantiated with other rules, some of which result in deterministic versions of the scenario. They can be used to throw light on the lottery. Here are some simple options for R:

R_1. Each god removes the highest-numbered ball passed to him

R_2. Each god removes the lowest-numbered ball passed to him

R_3. Each god removes the second-lowest-numbered ball passed to him

All three rules are deterministic. They are also all "locally consistent", in the sense that for *each* god it is consistent to follow the instructions using that rule, irrespective of what previous gods have done. Option R_1 is also "globally consistent": *all* the gods can follow that rule without contradiction. Indeed, option R_1 simply results in the ball numbered 1 remaining.

However, option R_2 is globally inconsistent. There is no way the scenario can play out with all the gods following that rule. And, even though option R_3 is biased more towards low numbers than high numbers, it is globally inconsistent as well. Based on these examples, it would seem that a rule has to be *strongly* biased in favor of low numbers to be globally consistent.

We can confirm this conjecture, while also making it precise, by setting up a formal framework for describing the space of all deterministic rules. The most obvious way to do this is to use a function for which the function value of any $n \in \mathbb{N}$ is the set of those balls that are left in the collection by god n, as given by the numbers with which they are marked. However, it will prove more interesting to use a type of description in which each god's choice is given relative to what he receives, in line with the three informal rule descriptions above. A deterministic rule can then be identified with a *selection function* s from $\mathbb{N}$ to the power set of $\mathbb{N}$, such that for each $n \in \mathbb{N}$, $s(n)$ is an n-element subset of $\{1, \ldots, n+1\}$. Here, $s(n)$ is the set of those balls that god n leaves in the collection, as numbered by their relative position in the collection that he receives, not by the numbers with which they are marked. So, for example, $s(2) = \{1, 2\}$ means that of the three balls that god number 2 receives, he passes on the the lowest-numbered and the second-lowest-numbered, while removing the highest-numbered.

This family of selection functions can be used to explore the relationship between local and global consistency. Taking only local consistency into account, it would seem that every selection function corresponds to a course of action that the gods could take (for example, R_1, R_2, and R_3 can each be identified with such a function). However, because of global consistency, only some of them actually do. Let us characterize the set of those that do.

We begin with a simpler problem: what does it take for the winning number to be well-defined? (This is a necessary but not a sufficient condition for consistency.) Say that god 1 selects the lowest-numbered ball passed to him as the winning number, i.e., $s(1) = \{1\}$ or, equivalently, $\{1\} \subset s(1)$. It is then possible, given just this information, that that ball is numbered 1. The winning ball, if well-defined, is the lowest-numbered ball passed on by god 2. That is again identical to the lowest-numbered ball passed on by god 3 iff $\{1\} \subset s(2)$. It is also identical to the lowest-numbered ball left in the collection by god 4 iff $\{1\} \subset s(3)$. If all those conditions are met, it is still a possibility that this is ball 1. But say that god 4 removed the lowest-numbered ball passed on to him, i.e., $\{1\} \not\subset s(4)$. Then, the lowest possible value of the winning ball is 2. Moving further back, god 5 would have had to leave both the lowest-numbered and the second-lowest-numbered ball passed to him in the collection, i.e., $\{1, 2\} \subset s(5)$, for the lowest possible value of the winning ball to still be 2 after we have taken him into consideration; otherwise, it is 3.

We see, then, that the winning number is well-defined if and only if there are only finitely many gods that raise the minimal possible value of the winning number in this way. We can express this observation more formally by defining the function l_1 recursively as follows:

$$l_1(0) = 1 \qquad l_1(n+1) = \begin{cases} l_1(n) & \text{if } \{1, \ldots, l_1(n)\} \subset s(n+1) \\ l_1(n) + 1 & \text{if } \{1, \ldots, l_1(n)\} \not\subset s(n+1) \end{cases}$$

The winning number is well-defined iff there is an upper bound to the values of $l_1(n)$ for $n \in \mathbb{N}$.

We can identify a similar criterion for whether the penultimate ball—the ball that makes it until god 1, but is discarded by him—has a well-defined number. It is well-defined iff there is an upper bound to the values of $l_2(n)$ for $n \in \mathbb{N}$, where l_2 is defined as follows:

$$l_2(1) = 2 \qquad l_2(n+1) = \begin{cases} l_2(n) & \text{if } \{1, \ldots, l_2(n)\} \subset s(n+1) \\ l_2(n) + 1 & \text{if } \{1, \ldots, l_2(n)\} \not\subset s(n+1) \end{cases}$$

In general, we can define l_k for all $k \in \mathbb{N}$ as follows:

$$l_k(k-1) = k \qquad l_k(n+1) = \begin{cases} l_k(n) & \text{if } \{1, \ldots, l_k(n)\} \subset s(n+1) \\ l_k(n) + 1 & \text{if } \{1, \ldots, l_k(n)\} \not\subset s(n+1) \end{cases}$$

The entire process conducted by the gods is well-defined iff there is an upper bound to the values of $l_k(n)$ for $n \in \mathbb{N}$ for all $k \in \mathbb{N}$.[3]

We have now shown that a very strong bias in favor of low numbers is needed for a deterministic rule to be consistent. This is also relevant when considering stochastic rules, for the possible outcomes of running the scenario with the gods following a given stochastic rule are some or all of the outcomes that the deterministic rules result in. (Here and until the final section, "outcome" means the entire sequence of events during the supertask, as described by a selection function, and not just a winning number.)

[3] The maximum of $\{l_1(n) | n \in \mathbb{N}\}$ is the winning number, if it exists. The maximum of $\{l_2(n) | n \in \mathbb{N}\}$ is not necessarily the value of the penultimate ball; it is the value of the highest-numbered ball that is among the last two; and similarly for l_k for all $k \geq 2$.

If a stochastic rule has among its possible outcomes one or more of the selection functions that describe an inconsistent deterministic rule, then that stochastic rule is inconsistent. And that is the case for the stochastic rule

R_4. Each god n chooses randomly and independently which ball to remove according to a uniform distribution on $\{1, \ldots, n+1\}$ and a relative numbering of the balls

which corresponds to both the specific scenario described in section 1 and to Kerkvliet's constraint. According to rule R_4, *every* selection function is a possible outcome,[4] so rule R_4 is inconsistent.

4 Pérez Laraudogoitia's coin-tossing supertask

Having failed to design a fair infinite lottery by removing balls from a collection, the gods instead try to do it by tossing coins, as suggested by Pérez Laraudogoitia (2014). Each god will toss a coin, and they will do it in the same reverse order as before: god n will toss a coin at $t = \frac{1}{n}$. They intend to use this stochastic rule:

R^1. (i) Each toss is such that the probability of tails and the probability of heads are both $\frac{1}{2}$, independently of other tosses. (ii) A dollar coin is used iff no previous toss made with a dollar coin came up tails; otherwise a euro coin.

Following Pérez Laraudogoitia, the gods infer the following from part (ii) of rule R^1:

(A1) If, in a given tossing of the dollar coin, tails comes up, the dollar coin shall not be tossed again. In particular, it is not possible for two tossings of the dollar coin to give tails as a result. If at $t = \frac{1}{n+1}$ the dollar coin is tossed and comes up heads, the dollar coin will be tossed again at $t = \frac{1}{n}$.

(A2) The dollar coin cannot be tossed only a finite number of times (zero included).

(A3) For any positive integer n, if the dollar coin is not tossed at $t = \frac{1}{n+1}$, then it will not be tossed at $t = \frac{1}{n}$ either.

The gods therefore believe that they can have a fair lottery on $\mathbb{N}_0$ by letting the winning number be the number of the unique god who gets tails using

[4] Even though, of course, under rule R_4 each selection function has probability 0.

the dollar coin, or, if none of them does that, 0: those options exhaust the possibilities, according to the above; and each one has probability 0, as can be inferred from part (i) of rule R^1 (see Pérez Laraudogoitia's paper).

However, the gods are mistaken. R^1 gives the appearance of being a rule the gods can follow, because it is locally consistent for each god, but it is globally inconsistent. I will explain why in two different ways: first in a simple way, by comparing it with a specific deterministic rule, and then through more formal considerations. The deterministic rule is the following:

R^2. (i) Even-numbered gods place their coin heads up, and odd-numbered gods place their coin tails up. (ii) A dollar coin is used if and only if no previous toss made with a dollar coin came up tails; otherwise a euro coin.

This rule is globally inconsistent. However, its second clause is identical to the second clause of R^1, and its first clause specifies something that must be a possibility according to the first clause of R^1. Therefore, R^1 is globally inconsistent.

The second way to explain it is by using a formal outcome space. When the gods can use either a dollar coin or a euro coin, the outcome space is some subset of $\{DH, DT, EH, ET\}^{\mathbb{N}}$, where DH stands for the dollar coin being tossed and heads coming up, and similarly for DT, EH, and ET. Each element of the outcome space is an omega-sequence. I will write it in the opposite order of what is standard, in order to reflect the temporal order. So, for example, the outcome in which god 1 tosses a tails with the dollar coin after every other god got a heads with the dollar coin can be denoted $(\dots, DH, DH, DH, DT)$.

Part (i) of rule R^1 implies that for every element $(\dots, a_3, a_2, a_1)$ of $\{H, T\}^{\mathbb{N}}$, there is an element $(\dots, b_3, b_2, b_1)$ of $\{DH, DT, EH, ET\}^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$, if $a_n = H$, then $b_n = DH$ or $b_n = EH$; and if $a_n = T$, then $b_n = DT$ or $b_n = ET$, which is a possible outcome. "Possible outcome" here means that only the element of chance can prevent it from becoming the actual outcome; its status as possible cannot be defeasible in any other way, on pain of contradiction. So contradiction is exactly what we get from part (ii) of rule R^1: that part implies (via (A1), (A2), and (A3)) that *only* elements of the form $(\dots, DH, DH, DH, DT, E, \dots, E)$, where each E is either EH or ET, and the element $(\dots, DH, DH, DH)$ are possible outcomes. Ergo, it is again seen that R^1 is globally inconsistent.[5]

[5]In Pérez Laraudogoitia's original version of this supertask, the coin-tossing ceases after the first tails, rather than the first tails resulting in a dollar coin being replaced with a euro coin. I changed this detail to ensure that there is a way to translate the informal requirement that each coin toss is fair and independent into a formal statement in a way that is neutral between Pérez Laraudogoitia's position and mine. I hope it is obvious

5 Assessment of the dialectical situation

An option that should be taken into account is that the two generic scenarios *as such* are impossible, and not only those instances thereof that we have already shown to be inconsistent. One might suspect as much, given that constraints that cannot be accounted for by causality are needed to uphold consistency. (No causal explanations can be given for why the gods have to be biased towards low numbers or for why they cannot follow rule R^1.) This impossibility could be due to a logical inconsistency not yet discovered, or it might be that metaphysical possibility is more restricted than logical possibility. And the impossibility could apply to these (generic) scenarios, or to reverse supertasks in general, or even to all supertasks.[6]

What I have shown in this paper is that even if reverse supertasks are possible, the attempts that have been made in the literature so far to utilize them to construct fair infinite lotteries have failed. And to the best of my knowledge, there have been no suggestions for concrete mechanisms for effecting fair countable lotteries that do not involve reverse supertasks. Consider, in that light, this quote by Howson (2014, p. 991):

> [T]here seems to be no reason in principle why anyone should not have an evenly-distributed belief over a countably infinite partition—they might, to take a fanciful example, think that some occult agency had rigged a lottery to give equal chances to each number

It would seem that there is no way to spell out that example. There is no particular method the agency could use to rig the lottery. The rigging would have to happen in a single, unanalysable act. That indeed makes the agency occult and the example fanciful. Given that, it does not seem plausible that it should be considered rational to assign credences to a countable outcome space in a uniform way.

However, it should be pointed out that the aim of this paper has only been to provide a certain kind of negative justification for the proposition that it is irrational to assign uniform credences to a countable outcome space. I have not considered the many general arguments pro and contra mere finite additivity; I have just undermined what would otherwise have been some support for my opponents' thesis. And even that undermining is potentially

enough that this change facilitates clarity without altering the scenario in any substantial respect: the results of the tosses with euro coins do not affect any other tosses (neither whether a dollar or euro coin is to be used, nor the probabilities of tails and heads) just like the absence of a toss does not affect any other toss (or whether there should be a toss at any given instant of time).

[6] In the last case, it would have to be false, contrary to what Zeno thought, that running a finite but positive distance involves infinitely many constituent actions.

defeasible: someone just have to come along and describe a mechanism that works.

6 Acknowledgments

The reverse supertask of section 1 is, essentially, of my invention. I first presented it in an unpublished paper, from which Timber Kerkvliet learned about it. In my original version, each god halved the number of balls in the collection. Timber Kerkvliet found it simpler to work with the version in which each god just removes a single ball. I also realized that the formalism of this paper would be more elegant with that version, and I therefore adopted it. I have benefitted from discussions about this supertask with Jon Pérez Laraudogoitia, Øystein Linnebo, Martin Jullum, Phil Chodrow, Laureano Luna, Carl Baker, Federico Luzzi, Thomas Brouwer, Andreas Fjellstad, Crispin Wright, John Norton, and Timber Kerkvliet.

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