Brouwer's Lawless Choice Sequences

Casper Storm Hansen

Draft of August 16, 2018

Abstract: According to L.E.J. Brouwer, there is room for non-definable real numbers within the intuitionistic ontology of mental constructions. That room is allegedly provided by lawless choice sequences, i.e., sequences created by repeated random choices of elements by a creating subject in a potentially infinite process. Through an analysis of the constitution of free choice sequences, it is the purpose of this paper to argue against Brouwer's claim.

Keywords: Intuitionism, Potential infinity, Choice sequences, The continuum, Platonism

According to (most non-formalist accounts of) classical mathematics, there are real numbers that cannot be individually defined. Prima facie you would think that if you reject platonism and actual infinity and believe that mathematics must be based on what can be constructed by the mind, then you would also have to reject these non-describable real numbers. Brouwer disagrees. He claims that non-definable real numbers are available in the form of choice sequences that are constructed in a potentially infinite process that is random, in the sense that the creator refrains from following a law for the construction. The purpose of this paper is to argue that he is mistaken.

I will start out with a general (and uncritical) introduction to intutionism that goes beyond what is strictly needed to get to the specific issues I will discuss critically.[1] I feel that this is necessary because intuitionism is too often thought of as the *result* of applying a non-standard logic, when instead it is based on metaphysical considerations;[2] and those metaphysical considerations (somewhat confused, as I will argue that they are) are more important to the issue of arbitrary real numbers than matters of logic. This introduction, in section 1, will be redundant for the Brouwer expert, but should be useful for the reader who primarily know Brouwer's ideas through logic-centered work on intuitionism. Section 2 lays out the specific target claim of this paper, which is then discussed in sections 3 and 4.

[1] See (van Atten 2004) and (van Stigt 1990) for more thorough introductions.

[2] This putting the cart before the horse has happened in spite of Brouwer's often repeated claim (see, e.g., his (1907, chapter 3), (1947) and (1952)) that mathematics is independent of, and primary to, language and logic.

1 General introduction

L.E.J. Brouwer, the father of intuitionism, aimed for a mathematics that avoids abstract objects and actual infinity. He does so by identifying the subject matter of mathematics with the potential infinity of mental constructions of a creating subject. Inspired by Kant (1781), Brouwer ontologically locates mathematics in the human intuition of time.[3] The basic building block of mathematical constructions is the so-called empty two-ity, which is the result of fixing on a moment of time, noticing it giving way to another moment of time, and abstracting away the contingent and specific elements of the experience that the subject happens to have at that moment. The construction of the empty two-ity gives us the numbers 1 and 2. That can be iterated by dividing the *now* of the initial two-ity's *past-now* distinction into a "new past" and a "new now" moments, resulting in an object *old past-(new past-new now)* that can play the role of the number three, and so on. According to Brouwer, the mathematical universe is limited to what can be constructed in this way.[4]

Brouwer can account for the meaning and truth of "$2+2=4$" as follows: I have constructed a two-ity, then another two-ity and then a four-ity, and succeeded in constructing a bijection between the disjoint union of the two former and the latter. That account is in terms of actual constructions. To account for the necessity of the truth of "$2+2=4$" and to account for the truth of "$10 \cdot 10^{100} = 10^{101}$" we have to go beyond actual constructions, but we can do that while staying within the confines of intuitionism. While I may make a mistake in an attempt to construct a truth maker for "$2+2=4$", a mental construction can come with an *intention* to execute the construction in a certain way and this intentionality implies that there is a normative aspect to constructions, which allows us to say that any *correct* construction of the sum of two and two would necessarily result in four. And though I will never actually construct the mental object that "$10 \cdot 10^{100} = 10^{101}$" is properly about *using* the intuition of time, *reflection* on the intuition of time shows the subject that the future is in principle (in some sense of "in principle") open-ended and that the series of natural numbers could therefore in principle be extended indefinitely. It is therefore clear that even the enormous numbers referred to in this sentence are potentially constructable, and that suffices (because we can prove in advance that if they were constructed correctly, then they would relate in the way indicated by the sentence). Thus, Brouwer's mentalism provides support for a mathematics of potential infinity but implies a rejection of actual infinity.

According to Brouwer (1908), the mentalistic ontology also necessitates a rejection of classical logic. A simple illustration can be given with the classical proof that there exist irrational numbers a and b such that a^b is rational. It is a proof by cases: Either $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. If it is rational, let both a and b be equal to the irrational number $\sqrt{2}$, and then a^b is rational.

[3] See page 8 and chapter 2 of (Brouwer 1907).

[4] "In this way" is very vague. While some of the details will be filled out below, the phrase also reflects a vagueness and lack of details in Brouwer's own papers. See (Kuiper 2004) for an attempt at filling out some of the details omitted by Brouwer.

If it is irrational, let a be equal to $\sqrt{2}^{\sqrt{2}}$ and let again b be equal to $\sqrt{2}$, in which case we have

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2,$$

i.e. again a rational number. This proof is non-constructive, as it does not inform us which irrational number a has the sought-after property. And for Brouwer that is an epistemic point with ontological implications: if we have not constructed an irrational number a and constructed its having the property of being equal to a rational number when raised to the power of an irrational number b, then there is no such number, for there is nowhere else in all of Being to locate it than in our constructions.

The culprit in the classical proof is the very first step, the assumption that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational in the absence of a construction to support one of the disjuncts. Thus *tertium non datur* is not in general a valid principle.

For Brouwer, logic does not have the central position in mathematics that it has according to the classical mathematician. Logical laws are merely highly general descriptions of the interrelations of constructions. Actually, they are merely highly general descriptions of the *language* that can, imperfectly, be used to convey an essentially language-less construction from one subject to another. An inference rule being valid means that whenever constructions corresponding to the premises are at hand, a construction corresponding to the conclusion can be effected.[5]

The non-standard ontology in general and the revision of logic in particular mean that a long range of important classical theorems fail intuitionistically. Another example is the theorem that every real number is positive or non-positive, $\forall x{\in}\mathbb{R}(x > 0 \lor x \leq 0)$. Brouwer gives examples of real numbers for which we cannot assert that it is one or the other.

A prerequisite for these examples is the intuitionistic notion of real numbers. With the exception of the strict finitist, all parties to the debate agree that a real number is an infinitary object. Either it is an ordered pair of actually infinite sets of rational numbers (Dedekind 1872), an actually infinite equivalence class of actually infinite, converging sequence of rational numbers (Cauchy 1821; Heine 1872), or, if you ask Brouwer, a potentially infinite, converging sequence of rational numbers. A real number is the process of a creating subject constructing more and more elements of a so-called free choice sequence. The elements can be freely chosen by the subject, or she can decide to follow a rule when choosing elements. In the latter case, it must be possible to calculate each element in a finite amount of time for which an upper bound is known in advance. The specific details of the definition of "real number" can be filled out in several different, intuitionistically acceptable ways. For the purpose of this paper, let us define a real number as a free choice sequence $\langle q_1, q_2, q_3, \ldots \rangle$ of rational numbers, such that $|q_n - q_{n+1}| \leq 2^{-n}$ for all natural numbers n.

[5] For more in this, see (Hansen 2016).

A real number for which it can neither be asserted (at present) that it is positive nor that it is non-positive can be constructed using a so-called fleeing property, defined by Brouwer (1955, 114, original emphasis) as follows:

> A property f having a sense for natural numbers is called a *fleeing property* if it satisfies the following three requirements:
>
> (i) For each natural number n, it can be decided whether or not n possesses the property f;
>
> (ii) no way is known to calculate a natural number possessing f;
>
> (iii) the assumption that at least one natural number possesses f, is not known to be contradictory.

An example of a fleeing property P is, for a given finite sequence of digits not yet found in the decimal expansion of π and not yet proved not to occur in it, that that sequence occurs beginning at the n'th decimal. Then let the real number a be defined as the free choice sequence the begins with the elements $1/4, -1/8, 1/16, \ldots, (-1/2)^{n+1}, \ldots$, and continues like that as long as no n has had the property P, and stays constant at $(-1/2)^{n+1}$ from the first n that has the property P onwards (if such an n is found). Then at any given point in the construction where the choice sequence is still "oscillating", the creating subject is not in possession of a truth maker for either of the sentences $a > 0$ and $a \leq 0$.

This invalidity of a classical theorem leads to the validity of a non-classical theorem, namely that all functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous (Brouwer 1924). Let me illustrate by explaining why this is an illegitimate definition of such a function:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

The function would have to map a to a free choice sequence $f(a)$. The first two elements of $f(a)$ could both be equal to $\frac{1}{2}$, for that is consistent with subsequent elements of $f(a)$ converging to 0 and consistent with subsequent elements of $f(a)$ converging to 1. However, as we cannot make it the case that $a > 0$ or $a \leq 0$ with a finite calculation with a pre-known upper bound on time consumption, there is no way to choose a third element of $f(a)$, for any possible choice would either be too far away from 0 or too far away from 1 to make it possible to have the sequence converge to that value if a subsequently attains a specific value (because a natural number is determined to have the property P or it is determined that it is impossible that any natural number does). Thus f is not a *total* function on the real numbers, but only a partial function defined for those real numbers that are either positive or non-positive.[6]

[6] This was merely an illustration, meant to convey an intuitive understanding. It does not qualify as an outline of a proof, for then it would have to conclude by an application of double negation elimination, which is also intuitionistically invalid. A proper proof of the theorem proceeds from the Fan Theorem which is a corollary of the Bar Theorem. The simplest self-contained proof of the continuity theorem in the literature is, as far as I know, to be found in (Heyting 1956).

Another way to create a real number that may be neither positive nor non-positive is, according to Brouwer, via a lawless choice sequence. Lawless choice sequences and the closely related subject of arbitrary real numbers are the specific elements of intuitionism that I want to focus on.

2 Brouwer on lawless choice sequences

The one aspect of Brouwer's intuitionism that distinguishes it most from other types of constructivism is his use of choice sequences. A choice sequence is a sequence that is created in time by successive choices of new elements by a creating subject (Brouwer 1952, 142). Only a finite initial segment has been constructed, at any point in time. The sequence, therefore, is never finished, but always in a state of expansion. According to Brouwer, by basing mathematics on such objects, the need to assume that something actually infinite exists is avoided.

The subject can choose to pick the elements according to an algorithm; for example an algorithm that selects rational numbers which are increasingly better approximations to π. That brand of choice sequences are called lawlike sequences. The opposite extreme consists of lawless sequences where each choice of an element is made at random. The individual may grant herself the freedom of allowing each element to be *any* member of some species[7], e.g. the species of natural numbers, or she may elect, from the beginning of the construction or at any point during it, to impose restrictions on her own future choices. As long as these restrictions allow for multiple different options, we can, for present purposes, categorize the sequence among the lawless. An important example is the decision to create a real number. This amounts to the subject imposing on herself the restriction that each element shall be a rational number q_n satisfying $|q_{n-1} - q_n| \leq 2^{-(n-1)}$ if $n > 1$.

According to the platonist, there are real numbers that cannot be defined among the abstract mathematical objects. While disagreeing with classical mathematics in many other respects, including whether abstract objects exist, Brouwer also claimed to have found a place for undefinable real numbers in the intuitionist ontology, namely among the lawless choice sequences. This thesis is perhaps presented most clearly in this quote:

> [Intuitionism] also allows infinite sequences of pre-constructed elements which proceed in total or partial freedom. After the abandonment of logic one needed this to create all the real numbers which make up the one-dimensional continuum. If only the predeterminate sequences of classical mathematics were available, one could by introspective construction only generate subspecies of an ever-unfinished countable species of real numbers which is doomed always to have the measure zero. To introduce a species

[7] A species is the intuitionistic counterpart of a set: an intensional collection of finitely or potentially infinitely many objects, see e.g. (Brouwer 1948, 1237).

> of real numbers which can represent the continuum and therefore must have positive measure, classical mathematics had to resort to some logical process, starting from anything-but-evident axioms[...] Of course, this so-called complete system of real numbers has thereby not yet been created; in fact only a logical system was created, not a mathematical one. On these grounds we may say that classical analysis, however suitable for technology and science, has less mathematical reality than intuitionist analysis, which succeeds in structuring the positively-measured continuum from real numbers by admitting the species of freely-proceeding convergent infinite sequences of rational numbers and without the need to resort to language or logic. (Brouwer 1951, 451–452)

So the claim is that the free creation of sequences—an arbitrary choice of an element, followed by another arbitrary choice of an element, *ad infinitum in potentia*—can result in sequences that cannot be defined. Without relying on abstract objects, but just the human potential for free mental construction, the intuitionist has access to the "full" set of real numbers.[8] This is the claim I want to dispute.

3 Constitution of free choice sequences

So much for introductory explanations. We shall now turn on the critical sense and try to get a more precise answer to the question of what a lawless choice sequence is. What exactly constitutes it?

As is witnessed by the debate on personal identity, questions of constitution can often be elucidated by first asking about the related questions of individuation and self-identity over time. So, if I begin a lawless sequence

[8]There are two slightly different ways to interpret the claim. The stronger interpretation is that Brouwer does, in one crucial respect, exactly the same as the classical mathematician, namely finding a non-denumerable totality of points with which to *identify* the continuum. If so, Brouwer has changed his mind in his late work. Because in his early years, before he came up with the idea of lawless choice sequences, Brouwer was of the opinion that the continuum is a primitive notion; that it cannot be constructed out of entities of another type; and that, specifically, it can not be identified with a set of points. He describes the continuous and the discrete as complementary and equally basic aspects of the Primordial Intuition (Brouwer 1907). Points (and numbers) could only be used to analyze a pre-given continuum by being the endpoints of the subintervals into which it can be decomposed. (This is, of course, the view originating with Aristotle (1930).) One reason he gives for why a continuum cannot be a set of points is that the available points are only those that can be identified with rational numbers or definable real numbers, i.e., lawlike sequences, which implies that there is only a denumerable infinity of them and hence not enough to exhaust the continuum (Brouwer 1913). However, it is also possible to interpret the quote in a weaker way: instead of the claim being that his reals make up the one-dimensional *intuitive* continuum, they just make up the *mathematical* continuum, i.e., they make up the best possible model we can have of the intuitive continuum. That is consistent with this model falling short of being a perfect model. With this interpretation, Brouwer makes a more modest claim, namely that the lawless sequences adds to the model something which the lawlike sequences cannot accomplish. The subtle differences between these exegetical thesis do not affect the critique made below.

of natural numbers now at t_1 by making the first element 4, and then *now* at t_2 add 9 to it as its second element, what is it that makes the sequence at t_1 identical to the sequence at t_2?

The strongest possible answer, that they are qualitatively identical, can quickly be ruled out. If they were qualitatively identical they would have exactly the same properties, so if they were qualitatively identical, then the sequence should have the property at t_1 that it has 9 for its second element, as it has that property at t_2. So, by the same token, it would be the case for each n that at t_1 it would be a property of the sequence that there was some specific number that was its nth element. Then the sequence would be actually infinite.

Instead of the property being *has 9 for its second element*, it could be *has, at t_2 and later, 9 for its second element*. But this makes little difference because the problem still arises, *mutatis mutandis*, in that there are still an actual infinity of properties. The fact that some of them are *about* the future does not make for a relevant difference. Brouwer and those of us who reject actual infinity can not accept that what will happen in the future corresponds, in general, to facts in the present. That is, not when the assumption of the possibility of an infinite future with genuinely random events is made, and Brouwer needs that premise for choice sequences to play the rôle of non-definable real numbers. Hence, he is committed to anti-realism with respect to the future.

The failure of this attempt to reach a satisfactory answer teaches us two things: that we must look for some criterion of numerical identity instead, and that this criterion must allow for the sequence to be genuinely dynamic in nature. This is acknowledged by Brouwer (1955, 114) who wrote that:

> In intuitionist mathematics a mathematical entity is not necessarily predeterminate, and may, in its state of free growth, at some time acquire a property which it did not posses before.

However, commenting on this quote, van Atten (2007, 14) states that:

> Observe that a property such as 'The number n occurs in the choice sequence x' is constitutive of the identity of x, but is generally undecidable and does not satisfy PEM [principle of the excluded middle].

If this were true, the property *the number 9 occurs in the choice sequence α* would be constitutive of α, which implies that the t_1-incarnation of α is not α. Consequentially, diachronic self-identity of a choice sequence would be impossible. At most, it can be the case that the property *the number n occurs in the choice sequence x* is constitutive of the identity of x from the point of time when n is added to the sequence. On pain of commitment to actual infinity, it cannot be before. And from that time onwards, it is decided.[9]

In order to avoid actual infinity in both its explicit and implicit forms, do we need to conclude that the temporal instantiation of our lawless sequence at t_2 is the object

$$\langle 4, 9 \rangle?$$

No, for that is just an ordered tuple, and a choice sequence is obviously not just *that*. There is a dynamical aspect to a sequence which is lacking from the n-tuple. This difference is, however, not in the past; also the tuple has been created, one element added at a time, in a temporal process. In Brouwer's universe there are no atemporal mathematical objects,[10] it is just that some of the temporal objects have been completed. That is the difference between the tuple and the sequence: the former has found its final form while the latter will continue to undergo changes.

This is, however, exactly the kind of claim that we have to be cautious about interpreting. The fact that it "will continue to undergo changes" must not be understood as an assertion about the actual future of the sequence, for the actual future does not exist. Given the commitment to anti-realism with regard to the future, the only content this claim can have is that the creating subject has an *intention* to amend the sequence. So, allowing "intention to expand" to be short for "intention to expand according to the restriction ..." if there is a restriction, this is a more promising proposal as to the constitution of our lawless sequence at t_2:[11]

$$\langle 4, 9, \text{intention to expand} \rangle$$

According to this answer, the present product of an ongoing construction is merely what has actually been constructed plus the psychological fact that its creator does not consider it finished. The self-identity of the sequence over time does not rely on any objects in the future, but simply on the subject choosing, when she adds a new element, to consider the extended sequence identical with the old one.

[9] Reacting to a draft of this paper, van Atten has informed me that he only intended to say that if the third element of α has been chosen to be 1, then it is known that a choice sequence β, for which something different from 1 has been chosen as its third element, is not equal to α.

[10] "[M]athematics [has] its origin in the basic phenomenon of the perception of a *move of time*, which is the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory." (Brouwer 1954, 2, original emphasis)

[11] This may be a simplification. Brouwer has been interpreted in phenomenological terms, according to which we do not experience extensionless points of time (see (van Atten 2007, 33–34) and (Becker 1923)). Elements are thus chosen during intervals of time rather than at instants of time. In addition to earlier elements being kept in retention and recollection, to use Husserl's (1964) terminology, the next element or the next few elements may be anticipated in protention. But the slight vagueness that this may introduce does not substantially influence the points that follow, precisely because there can only be a limited number of specific, individually chosen future elements which can be within the scope of protention. If an infinite number of elements is anticipated, it can only be in the form a rule, or simply as the anticipation of continuing to make choices (i.e. without the specific choices being part of the anticipation).

I think it is the correct answer. By that I mean that this is the closest thing we can find in "the inventory of the world" to what Brouwer envisions a lawless choice sequence to be. Below it will be useful to contrast this answer with another possible answer, namely that the constitution of our choice sequence at t_2 looks like this:

$$\langle 4, 9, x_3^{\mathbb{N}}, x_4^{\mathbb{N}}, x_5^{\mathbb{N}}, \ldots \rangle$$

Here $x_n^{\mathbb{N}}$ is supposed to be an *indeterminate element* that is restricted to $\mathbb{N}$. That is, at t_2 it is true of, e.g., the third element that it is a natural number, but neither true nor false that it is equal to 7. Then at t_3 the choice sequence may change to

$$\langle 4, 9, 7, x_4^{\mathbb{N}}, x_5^{\mathbb{N}}, x_6^{\mathbb{N}}, \ldots \rangle,$$

as the next choice determinates the third element, which was until then indeterminate.

I think that $\langle 4, 9, \text{intention to expand} \rangle$ is the correct answer to the question of the constitution of the choice sequence at t_2 because it captures all of what seems to be the facts of the situation under a parsimonious ontological analysis thereof: 4 has been chosen as the first element, 9 has been chosen as the second element, and the creating subject has an intention to continue expanding the sequence; that's it. It is a simple situation and there is no need to invoke the existence of mysterious indeterminate objects to understand it.[12] (Hence, let us refer to it as the "simple answer" and to the alternative answer as the "indeterminacy answer".) However, I will consider both of these answers to the question of the constitution of lawless choice sequences in the following.

4 Evaluation of Brouwer's claim

Let us evaluate Brouwer's claim that he has succeeded in supplying an adequate ontology for the "full" system of real numbers that also includes non-definable sequences of rational numbers in the light of the analysis of the constitution of a free choice sequence.

A preliminary point is that at any given time, only a finite number of choice sequences actually exist, because a choice sequence only exists if someone

[12] I am overstating the simplicity a bit, even if we set aside the point in footnote 11. If you and I both produce a choice sequence and we have so far, by chance, picked the same elements in the same order, and we both intend to expand our respective sequences according to the same restrictions, if any, we have nevertheless produced different sequences. (The two sequences will be *equal* (so far), but not be *identical*. Brouwer also makes this distinction, for example in his definition of "species": "properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be equal to it, relations of equality having to be symmetric, reflexive and transitive" (1952, 142). Troelstra (1977) makes the distinction using the terminology "extensional identity" and "intensional identity".) That is not captured by "$\langle 4, 9, \text{intention to expand} \rangle$"; there are also concrete facts about who the creator of the sequence is, when it was started, etc., that belong in a complete analysis of the constitution of a choice sequence. However, this complication is irrelevant to the issue at hand: there is still no need to invoke the existence of indeterminate objects.

has created it. Thus, relying only on actually existing choice sequences will definitely not suffice. The idea has to be that $\mathbb{R}$ consists of all possible free choice sequences. And that is the idea: Where the classical, platonistic reals by virtue of the hierarchical nature of the set theoretical universe must all exist for the set of them to do the same, Brouwer only commits himself to the *possibility* of constructing each of his reals. They do not all have to exist prior to them being collected in the species of all reals. His continuum is the totality of all possible convergent sequences of rationals. I will not take issue with that. The question is: which free choice sequences are possible?

Assume that a is a platonic real number, i.e., that a is an actually infinite (and converging) sequence of rational numbers $\langle a_1, a_2, \ldots \rangle$, and assume further that this sequence is non-definable. If a creating subject attempts to construct the same ("same" in a mathematical, but not an ontological sense) real number, it is possible for her to construct $\langle a_1, \text{intention to expand} \rangle$, and then it is possible to expand that to $\langle a_1, a_2, \text{intention to expand} \rangle$, and then to $\langle a_1, a_2, a_3, \text{intention to expand} \rangle$. However, at each instant, only a finite initial segment of a has been created.

The actually infinite, non-definable sequences (assuming for the moment that they exist) do not correspond to possible routes for potentially infinite choice sequences. For "possible" means "can be taken", and the *entire* route corresponding to a platonistic non-definable sequences can never be taken. Only initial segments of those sequences can ever be taken.

There is a nice metaphor of Posy's (1976, 98–99) we can make use of here. He likens choice sequences to the route of a bus traveling on a forking highway. The journey of the bus can be seen from different perspectives. First, there is the perspective of a passenger in the bus seated with his back to the driver so that he can only see the route already traversed. Second, there is the perspective of the bus driver which in addition to the knowledge of his passenger has an intention of where to travel from his present position. Third, and last, there is the perspective of a pilot looking down on the bus and the road system from a helicopter hovering above, seeing both the traveled path and the roads ahead. Given the rejection of actual infinity, there is no helicopter perspective. Actually infinite roads are no less actually infinite than completed infinite travels. The only legitimate perspectives are the passenger's and the driver's, and in the former case that is a finitely extensional perspective and in the latter the perspective is finitely extensional and finitely intensional. For the bus driver or the creating subject, there is an infinity of possibilities. But one must not conflate an infinity of possibilities with the possibility of infinity.

If that was a bit too metaphorical, the point can also be made more formally, either by a comparison to classical mathematics or by employing tense logic. Assuming classical mathematics, we can say that lawless choice sequences can only deliver the elements of $\mathbb{N}^{<\omega}$ (or $\mathbb{Q}^{<\omega}$, etc.), not the elements of $\mathbb{N}^{\omega}$ (or $\mathbb{Q}^{\omega}$, etc.).

With the notation of tense logic we can disambiguate what has been conflated by intuitionism. An intuitionist would say that when a creating subject is constructing a lawless choice sequence a of rational numbers, then

$\forall n{\in}\mathbb{N}\,\exists q{\in}\mathbb{Q}\,(a_n = q)$ is true. But that is not, in any straightforward way, true of the present. Taking a cue from Prior (1967), we can see that there must be an implicit "it will be the case that"-operator somewhere. Writing this operator as "F", we can disambiguate $\forall n{\in}\mathbb{N}\,\exists q{\in}\mathbb{Q}\,(a_n = q)$ as either $\forall n{\in}\mathbb{N}\,F(\exists q{\in}\mathbb{Q}\,(a_n = q))$ or $F(\forall n{\in}\mathbb{N}\,\exists q{\in}\mathbb{Q}\,(a_n = q))$. Only the former is true, but it is the truth of the latter that would be required for α to be (or become) a real number and not just an always expanding, but always finite sequence.[13]

When platonism and actual infinity have been rejected, there is no *sub specie aeternitatis* perspective under which the process of extending one finite sequence to another finite sequence again and again can constitute an omega-sequence.

The conclusion is that choice sequences cannot do the same work as the classical set of real numbers allegedly do. Does that conclusion change if we replace the simple answer with the indeterminacy answer? I think that, underlying Brouwer's claim, there is some vague intuition that it does:[14] each individual indeterminate element "ranges", in some "fuzzy" way, over all the natural numbers (or over the members of some other species)—it is not true that $x_4^{\mathbb{N}} = 76$, but it is also not false—so the infinitely many indeterminate elements collectively range over all sequences of natural numbers in a way that is not restricted by what can be defined. However, I don't see how that intuition can be substantiated. The points made in the case of the simple answer still stand: First, in the attempt to construct a, the creating subject can construct $\langle a_1, x_2^{\mathbb{N}}, x_3^{\mathbb{N}}, x_4^{\mathbb{N}}, \ldots\rangle$, and then expand it to $\langle a_1, a_2, x_3^{\mathbb{N}}, x_4^{\mathbb{N}}, x_5^{\mathbb{N}}, \ldots\rangle$, and then to $\langle a_1, a_2, a_3, x_4^{\mathbb{N}}, x_5^{\mathbb{N}}, x_6^{\mathbb{N}}, \ldots\rangle$, etc., but she never gets any closer to a itself. Second, also the tail of indeterminate elements must be considered

[13] The branch of formal intuitionism where I think it should have been most obvious that the conflation happens is in Beth's semantics (Beth 1964, 444ff.), which is exactly an attempt at capturing the semantics of sequences of choices.

[14] The intuition is vague as Brouwer's official account of mathematical ontology offers no support for the assumption of there being indeterminate objects. And he is (what would otherwise count as) quite explicit in his delimitation of the mathematical realm: only mental constructs are admitted and only those that can be introduced in accordance with one of the two "acts of intuitionism" (Brouwer 1952). The first act of intuitionism is the purification of mathematics, where everything that cannot be grounded in the intuition of time is exorcised. The intuition of time gives the subject the awareness of a difference in the form of the before-after relation, or in Brouwer's own words, the so-called Primordial Intuition of the empty two-ity. As explained in section 1, this can be translated into the numbers 1 and 2, and the number 3 can be created by holding one before-after relation in retention while distinguishing it collectively from a new "after". By repetition, the natural numbers can be constructed and so can any finite object or set of finite objects equipped with relations and operations in a way that is not much different from how it is done classically. However, the second act of intuitionism is the realization by the creating subject that he is not limited to already created mathematical objects. Rather, he is free to employ the Primordial Intuition in any way he likes in a temporarily unbounded "free unfolding of the empty two-ity". This is what opens up for free choice sequences: the subject can set out to make a potentially infinite sequence consisting of "mathematical entities previously acquired". The second act is what makes Brouwer's universe potentially infinite instead of finite. But, it is a potential infinity of Primordial Intuition-created entities. The second act does nothing to sanction new kinds of basic objects. It just allows for the open-ended addition and combining of more and more mental constructs.

a potential infinity, and that precludes the elements from being independent of each other in the way needed for them to collectively have a range that includes a non-definable sequence. There is no equivalent of the arbitrary platonic real number a even if we pretend to believe that there are such things as indeterminate elements.

Let me consider a possible objection. An intuitionist might bite the bullet and accept that no lawless choice sequence is the equivalent of a, but claim that that is because lawless choice sequences are so fundamentally different from the objects of classical mathematics that there is no direct correspondence—and then proceed to claim that the lawless choice sequences nevertheless "fill up the holes" in the continuum that remain after only the lawlike sequences have been poured into it.

While this defense seems contrary to the spirit of the Brouwer quote in section 2, the objector may try to draw some support from Troelstra (1977, section 2.5). According to him, a lawless choice sequence must be extensionally different (see footnote 12) from any sequence that is intensionally different. While Troelstra's claim only ranges over intuitionistic sequences, it doesn't seem too much of a stretch to say that if a lawless choice sequence cannot be extensionally identical to any other intuitionistic sequence, then it also cannot be extensionally identical to a platonic sequence (or, at least, it doesn't seem to much of a stretch if one, for the sake of argument, is sufficiently eclectically minded to allow for such comparisons between intuitionistic and platonic objects).

This objection can be meet by reformulating the critique of lawless sequences. Instead of comparing them with classical sequences, we can instead point out that a lawless choice sequence cannot add anything to the constitution of the continuum that isn't already accomplished with rational numbers and lawlike choice sequences. Notice that the definition of "real number" given earlier implies that a real number is a convergent sequence of rational numbers where each element restricts all subsequent elements to an increasingly smaller interval around it, and each such interval must also be included in the previous intervals. So for a lawless choice sequence which is meant to be a real number (i.e., the creating subject restricts herself to choices that are in conformity with the definition), when n elements have been chosen, the first $n-1$ elements no longer carry any relevant information. For the nth element indicates which interval future choices are restricted to and all the earlier intervals include the nth interval and do therefore not restrict the creating subject any further. Therefore it makes no difference for the theory of real numbers if we identify the development

t_1: $\langle 1, \text{intention to expand}\rangle$,
t_2: $\langle 1, 1/2, \text{intention to expand}\rangle$,
t_3: $\langle 1, 1/2, 3/4, \text{intention to expand}\rangle$

with

t_1: $\langle 1, \text{intention to change}\rangle$,

t_2: $\langle 1/2, \text{intention to change}\rangle$,
t_3: $\langle 3/4, \text{intention to change}\rangle$.

At any given time, the mathematical content of a lawless sequence equals an interval with rational endpoints. The creating subject is just changing her mind about which interval to use, and each choice is one that could have been made initially. The implication is that lawless choice sequences do nothing that rational numbers cannot do.[15]

References

Aristotle (1930). Physica. In *The Works of Aristotle*. Clarendon Press. Edited by W. D. Ross.

Becker, O. (1923). Beitrage zur phänomenologischen Begründung der Geometrie und ihrer physikalischen Anwendungen. *Jahrbuch für Philosophie und phänomenologische Forschung 6*, 385–560.

Beth, E. W. (1964). *The Foundations of Mathematics* (2nd ed.). North-Holland.

Bishop, E. and D. Bridges (1985). *Constructive Analysis*. Springer.

Brouwer, L. E. J. (1907). *Over de Grondslagen der Wiskunde*. Ph. D. thesis, Amsterdam. Translated as "On the Foundations of Mathematics" in (Brouwer 1975).

Brouwer, L. E. J. (1908). De onbetrouwbaarheid der logische principes. *Tijdschrift voor wijsbegeerte 2*, 152–158. Translated as "The Unreliability of the Logical Principles" in (Brouwer 1975).

Brouwer, L. E. J. (1913). Intuitionism and formalism. *Bulletin of the American Mathematical Society 20*, 81–96. Reprinted in (Brouwer 1975).

Brouwer, L. E. J. (1924). Beweis, dass jede volle Funktion gleichmässig stetig ist. *Koninklijke Akademie van wetenschappen te Amsterdam, Proceedings of the section of sciences 27*, 189–193. Reprinted in (Brouwer 1975).

Brouwer, L. E. J. (1947). Richtlijnen der intuitionistische wiskunde. *Koninklijke Akademie van wetenschappen te Amsterdam, Proceedings of the section of sciences 50.*

Brouwer, L. E. J. (1948). Consciousness, philosophy, and mathematics. In *Proceedings of the 10th International Congress of Philosophy*, pp. 1235–1249. Reprinted in (Brouwer 1975).

Brouwer, L. E. J. (1951). Notes for a lecture. In *(van Stigt 1990).*

Brouwer, L. E. J. (1952). Historical background, principles and methods of intuitionism. *South African Journal of Science 49*, 139–146. Reprinted in (Brouwer 1975).

[15]Bishop's brand of constructivistic mathematics (Bishop and Bridges 1985) may be suitable for someone who wants to retain Brouwer's mentalism, Brouwer's rejection of actual infinity, and Brouwer's verificationism, while accepting this critique of Brouwer's intuitionism.

Brouwer, L. E. J. (1954). Points and spaces. *Canadian Journal of Mathematics 6*, 1–17. Reprinted in (Brouwer 1975).

Brouwer, L. E. J. (1955). The effect of intuitionism on classical algebra of logic. *Proceedings of the Royal Irish Academy 57*, 113–116. Reprinted in (Brouwer 1975).

Brouwer, L. E. J. (1975). *Collected Works 1*. North-Holland. Edited by A. Heyting.

Cauchy, A.-L. (1821). *Cours d'analyse de l'École royale polytechnique*. L'Imprimerie Royale. Translated as "Cauchy's Cours d'analyse: an annotated translation", Springer, 2009.

Dedekind, R. (1872). *Stetigkeit und irrationale Zahlen*. Friedrich Vieweg und Sohn. Translated as "Continuity and Irrational Numbers" in "Essays on the Theory of Numbers", Dover Publications, 1963.

Hansen, C. S. (2016). Brouwer's conception of truth. *Philosophia Mathematica 24*, 379–400.

Heine, E. (1872). Elemente der Funktionenlehre. *Journal für die reine und angewandte Mathematik 74*, 172–188.

Heyting, A. (1956). *Intuitionism: An Introduction*. North-Holland.

Husserl, E. (1964). *The Phenomenology of Internal Time-Consciousness*. Martinus Nijhoff.

Kant, I. (1781). *Critik der reinen Vernunft*. Hartknoch. Translated as "Critique of Pure Reason", P. Guyer, and A. Wood (eds.), Cambridge University Press, 1998.

Kuiper, J. J. C. (2004). *Ideas and Explorations: Brouwer's Road to Intuitionism*. Ph. D. thesis, Universiteit Utrecht.

Posy, C. J. (1976). Varieties of indeterminacy in the theory of general choice sequences. *Journal of Philosophical Logic 5*, 91–132.

Prior, A. (1967). *Past, Present and Future*. Oxford University Press.

Troelstra, A. S. (1977). *Choice Sequences*. Oxford University Press.

van Atten, M. (2004). *On Brouwer*. Wadsworth.

van Atten, M. (2007). *Brouwer Meets Husserl*. Springer.

van Stigt, W. P. (1990). *Brouwer's Intuitionism*. North-Holland.