Brouwer's Conception of Truth

Casper Storm Hansen
Department of Philosophy, University of Aberdeen, Aberdeen AB24 3FX, U.K.
E-mail: casper_storm_hansen@hotmail.com

Abstract: In this paper it is argued that the understanding of Brouwer as replacing truth conditions with assertability or proof conditions, in particular as codified in the so-called Brouwer-Heyting-Kolmogorov Interpretation, is misleading and conflates a weak and a strong notion of truth that have to be kept apart to understand Brouwer properly: truth-as-anticipation and truth-in-content. These notions are explained, exegetical documentation provided and semi-formal recursive definitions are given.

Consider the sequence $\gamma = \{c_n\}$, where c_n is defined to be equal to $(-1/2)^{k_1}$ if k_1 is smaller than n and the sequence 0123456789 appears for the first time in the decimal expansion of π with the 0 at the k_1^{th} decimal position, and equal to $(-1/2)^n$ if there is no such k_1. Then define r to be $\lim_{n\to\infty} c_n$.

Brouwer [1924a] claims that the proposition that r equals 0, is neither true nor false. It would only be true if we had a proof that there are no 0123456789-sequences in the decimal expansion of π, and it would only be false if we knew of such a sequence.[1] Assuming that it is true or false in the absence of such knowledge amounts to assuming that the decimal expansion of π has extra-mental existence.

When making such a claim, which notion of truth does Brouwer apply as his alternative to the rejected Platonic notion? The answer to this (purely exegetical) question is the goal of this paper. (Or perhaps more accurately: it is an attempt at a rational reconstruction.)

Before I present what I think is the correct interpretation of Brouwer, I will discuss and reject three other interpretations that have been made or could be made of his writings. I do not do this with a merely negative aim, but because my interpretation combines elements from those three and is, therefore, best understood and motivated by comparison with them. The first interpretation is that what is true can only be so because of what has actually been constructed. The second is that not only actual but also all potential constructions can serve as truth makers. (These first two options are quite naïve and are not serious contenders but they serve the purpose of stage setting.) And the third is that truth is equated with proof. This third

[1] We actually do today: there is a 0123456789-sequence beginning at decimal number 17,387,594,880 as well as at several other, later positions [Wells, 1986]. But as Brouwer [1951b] points out there will probably always be an ample supply of other examples that can be used instead. Hence, this is not really relevant, so we will just stick to Brouwer's example.

option has now become so entrenched as an interpretation of Brouwer that a word of warning is in order so as to forestall misunderstandings where it is read into the first two: the notion of proof plays no role in the first two interpretations and "construction" is not to be read as "construction of a proof". Rather, "constructions" are of the mathematical objects and relations between them that mathematical theories are about, not of the proofs and theorems about them.

1 Actual constructions, potential constructions and BHK

In several places Brouwer gives rather explicit answers to the question about what notion of truth he employs – answers which are nonetheless puzzling. Here is one:

> *[T]ruth* is only in *reality*, i.e. in the present and past experiences of consciousness. Amongst these are things, qualities of things, emotions, rules (state rules, cooperation rules, game rules) and deeds (material deeds, deeds of thought, mathematical deeds). But expected experiences, and experiences attributed to others are true only as anticipations and hypotheses; in their contents there is no truth. [Brouwer, 1948, 1243, emphasis original]

So Brouwer's official theory is that truth consists in correspondence with actual constructions. However, at first sight it does not seem that he adheres strictly to this credo. One thing that seems to be at odds with it is his claim that as soon as the subject has a decision procedure for a given proposition then *tertium non datur* holds for it. Sticking to actual constructions as truth-makers[2], one should presumably say that only when the decision procedure has been executed does the proposition gain a truth value. Another thing is that the theorems that he states are typically like classical theorems in that they cover an infinity of cases even though, obviously, not all these cases can have been realized as actual constructions. So it seems that Brouwer often relies on potential rather than just actual constructions.

If that is the case then it appears easy to explain why Brouwer has felt the pressure to do so. The claim that the "actualist" position commits you to, that the proposition that, say, the one millionth decimal of π is 1, was not true in Brouwer's time but has only become so since, is very weird. It is weird because, by Brouwer's own admission, it is determined in advance what the decimals of this lawlike sequence are.[3]

[2]In my analysis of Brouwer I will make extensive use of the concept of truth-makers, which was not available to Brouwer himself, as it only – as far as I know – dates back to [Mulligan et al., 1984].

[3]Brouwer [1948, 1237] also uses the word "predeterminate" for "lawlike". He also writes that the "freedom in the generation of [a free choice sequence] may at any stage be completely abolished [...] by means of a law fixing all future [terms] in advance" [Brouwer, 1954, 7].

So is the account of truth which Brouwer actually subscribes to, that whatever is determined in advance is true, i.e., that predetermined potential constructions are sufficient for truth? No, Brouwer does not go nearly far enough in this direction to warrant such an interpretation.[4] For it is also determined in advance whether there is a k_1, and if there is, what value it has. So by that standard it would be false, atemporally, that the limit of γ equals 0.

Framing the same point in terms of the popular example of Goldbach's Conjecture, it cannot be the case that it is fixed in advance for each n whether it would be a counterexample to the conjecture, but not fixed in advance whether such an n can be constructed. If we let P stand for the property of being a Goldbach number and n range over the even integers greater than 2, Brouwer claims that $\forall n(P(n) \vee \neg P(n))$ is true but that $\forall n P(n) \vee \neg\forall n P(n)$ is not. That difference cannot be accounted for if truth is a matter of predetermined potential constructions alone, for the two propositions are about the same potential constructions.[5]

Consistent reliance on potential constructions would make all truths about lawlike sequences timeless and independent of the subject's knowledge and would therefore be in conflict with the temporality of Brouwerian truth and in particular with the role played by possession of algorithms; when the subject acquires a means to "judge" a proposition, i.e., comes up with a decision procedure for it, *tertium non datur* becomes valid for it [Brouwer, 1952, 141].

Neither the actual existence of constructions nor the mere possibility of constructions can be made out to be Brouwer's criterion for truth. Rather, he seems to be somewhere in between, relying on potential constructions when the subject has knowledge of its finitude in advance and otherwise insisting on actual constructions. That difference cannot be explained with mentalism alone. So if this intermediate position is to be seen as more than an arbitrary compromise between conflicting sources of pressure, there must be some more fundamental truth criterion in play which can explain the unequal demands on what kind of existence of constructions is required.

Partly as an answer to this challenge, it has become common to interpret Brouwer as equating truth with existence of proof, or as it has also been formulated, to replace truth conditions with assertability conditions [Raatikainen, 2004]. There are certainly good textual reasons to believe that proofs play at least some role in Brouwer's conception of truth. For

[4]Even though he also commits explicitly to this second interpretation in writing, as he did for the first: in his own copy of his dissertation he changed "*bestaan* in wiskunde betekent: *intuïtief zijn opbebouwd*" to "*bestaan* in wiskunde betekent: *intuïtief op te bouwen*", that is, "*existence* in mathematics means: *to have constructed intuitively*" to "*existence* in mathematics means: *to be constructible intuitively*" [van Dalen, 2001, 134, footnote f, emphasis original].

[5]One way to analyze the concept of predetermination is with a counterfactual: if at time t_2 the nth decimal of π is found to be m, then for any $t_1 < t_2$ it would have been the case that if an agent had constructed the nth decimal at t_1, it would have been m. Does Brouwer deny this as the Kripkensteinian rule-following skeptic [Kripke, 1982] does? I do not think so. He just refuses to recognize such facts of predetermination as truths, presumably because he cannot locate a truth-maker for it within his anti-realist ontology.

one thing, the notion of proof is employed in some definitions of concepts which we would not normally consider to be about proofs (i.e., not a concept belonging to proof-theory):

> Two mathematical entities are called different, if their equality has been *proved* to be absurd. [Brouwer, 1952, 142, emphasis added]

A second reason is that the following two quotations are so alike that it is natural to interpret Brouwer as considering them nothing but rhetorical variations of each other even though one has "true" where the other has "proved to be true", suggesting equivalence between them:

> Correctness of an assertion then has no other meaning than that its content has in fact appeared in the consciousness of the subject. We therefore distinguish between:
>
> 1. true
> 2. impossible now and ever
> 3. at present neither true nor impossible
>   *a.* either with, or
>   *b.* without the existence of a method which must lead to either 1. or to 2. [Brouwer, 1951a]

> [I]n mathematics no truths could be recognized which had not been experienced, and that for a mathematical assertion a the two cases formerly exclusively admitted were replaced by the following four: 1. a has been *proved to be true*; 2. a has been *proved to be false, i.e. absurd*; 3. a has neither been proved to be true nor to be absurd, but an algorithm is known leading to a decision either that a is true or that a is absurd; 4. a has neither been proved to be true nor to be absurd, *nor do we know an algorithm leading to the statement either that a is true or that a is absurd*. [Brouwer, 1955, 114, emphasis original]

It is clear that proofs are in some way constitutive of Brouwer-truth. Nevertheless, it cannot be correct to interpret him as *identifying* truth and existence of proof. For that is exactly what he [1954] forcefully criticizes the formalists for doing; they render mathematics meaningless by doing away with the content that is being proved. The Brouwerian would ask rhetorically: if there is nothing beyond the proofs, then what is it that is being proved? Trying to reduce truth to proofs is to put the cart before the horse; there has to be something more basic that proofs can be about. To prove something must be to show that it is true. If there is no independent notion of truth, then the concept of proof is taken as basic, and intuitionism is very similar to formalism.

Also, insofar as existence of proofs is admitted as partially constitutive of truth in Brouwer's view, it must be with a careful understanding of what proofs are. They cannot be understood as linguistic entities and they cannot even be something that is "built" from logic, for mathematics is independent of, and prior to, language and logic, according to an often repeated claim of Brouwer's.[6]

An attempt at a more precise version of the interpretation of Brouwer as equating truth with existence of proofs is what has become known as the "Brouwer-Heyting-Kolmogorov interpretation" or "BHK interpretation" for short. It gives the meaning of the logical connectives and quantifiers by recursively stipulating what counts as a proof of a sentence: a proof of $\phi \wedge \psi$ consists of a proof of ϕ and a proof of ψ (and the conclusion); a proof of $\phi \vee \psi$ consists of a proof of ϕ or a proof of ψ; a proof of $\phi \rightarrow \psi$ consists of a method for converting any proof of ϕ into a proof of ψ; a proof of $\neg\phi$ consists of a method for converting any proof of ϕ into a proof of a contradiction; a proof of $\exists x \phi(x)$ consists of an object d, a proof that d is in the given domain and a proof of $\phi(d)$; and a proof of $\forall x \phi(x)$ consists of a method for converting any object d in the domain into a proof of $\phi(d)$.

(To this story must be added an account of what a proof of an atomic sentence is. Such accounts are specific to the mathematical theory under consideration. Arithmetic can be formalized in such a way that the only atomic sentences are numerical equations, and then a proof of such a sentence of the form $a = b$ can be specified as something that begins with identity statements of the form $a = a$.)

There are two specific problems for the BHK interpretation in addition to the already mentioned more general problems of the truth=proof interpretation. The first is that the interpretation of the disjunction is not faithful to Brouwer. He has it that *tertium non datur* already holds[7] for a proposition ϕ when the subject has a decision procedure for it, also prior to executing that procedure and thereby obtaining a proof of one of the disjuncts of $\phi \vee \neg\phi$. But the BHK interpretation does not allow us to assert that disjunction without being in a position to assert one of the disjuncts.

The second is, as Dummett [2000, 269–270] points out, that the definition, as it stands, is impredicative, because of the clauses for the conditional and the universal quantifier: A proof of $\phi \rightarrow \psi$ is a certain operation on all possible proofs of ϕ. We have no guarantee that we have a full grasp of what counts as a proof of ϕ before we have a full grasp of what counts as a proof in general, but that is just what is being defined.

I think the problem can be presented most forcefully in the form of a trilemma. Either (1) it is fixed in advance of the recursion on the complex sentences what counts as a proof of an atomic sentence or (2) it is not.[8]

[6] See, e.g., [1907, chapter 3], [1947] and [1952].

[7] In his [1952] Brouwer writes that in this case "application of the principle of the excluded third is *permissible*"; in his [1908] that it is "*reliable* as a principle of reasoning" (emphasis added).

[8] The case of arithmetic belongs, as explained, in the former category, but it is not clear where other mathematical theories belong.

The first case can be subdivided into a case (1a) where these prefixed proofs include some that contain, as lines in the proofs, complex sentences, and (1b) where they do not. In all three cases there are unacceptable consequences which can all be exemplified with a proof concluding with *modus ponens*, i.e. a proof where the antepenultimate line is ϕ, the penultimate is $\phi \to \psi$ and the final line is the atomic sentence ψ. In case (1a) this proof is valid independently of the BHK recursion, so the proof stripped of its last line is a proof of $\phi \to \psi$ independently of the BHK recursion, making it redundant. In case (1b) this cannot be a proof of ψ; so no atomic sentence can be proved by modus ponens, which is absurd. And in case (2) it is consistent with the BHK interpretation to stipulate that ψ can be proved from ϕ no matter what these sentences are, for we can just claim that the mentioned proof is a valid proof for ψ if we also claim that the method *to any proof of ϕ, add a line containing $\phi \to \psi$ and then a line containing ψ* is a proof of $\phi \to \psi$, for that is then the method for converting any proof of ϕ into a proof of ψ required by the BHK interpretation of the conditional.

Dummett [1978b, 2000] has tried to improve on this situation by distinguishing between "canonical proofs" and a weaker notion of proof. Canonical proofs are proofs that never proceed via formulae that are more complex than the premises and conclusion. On the other hand, informal proofs or "demonstrations", the kind of proofs that you typically find in a mathematical paper, are proofs that, in principle, provide a method for obtaining a canonical proof. A disjunction may therefore be assertable by virtue of a demonstration, which, when converted into a canonical proof, would not only prove the disjunction but also one of the disjuncts. The BHK interpretation is then taken to define the weaker notion of proof, while presupposing only the notion of canonical proofs, and then the recursion is well-founded.

However, even if this works as a solution to the specific problems for the BHK interpretation, it does not resolve the more general problems of interpreting Brouwer as identifying truth and existence of proof. (But, it should be noted, it was not intended as an exegetical thesis by Dummett.) Just splitting the notion of proof into two different notions of proof cannot do that job. The reason I have nevertheless discussed the idea is that it has similarities to the exegetical thesis I will propose. For, according to this, Brouwer effectually splits the notion of truth into a strong and a weak variant.

2 Two-truths interpretation

The key, I believe, is to be found in one of the quotations above, namely the first one in this paper: expected experiences are true only as anticipations; in their contents there is no truth. Here is a distinction between a strong notion of truth, truth-in-content, and a weaker, truth-as-anticipation. (I will use the abbreviation "TIC" for "truth-in-content" and also, ambiguously, for the adjective "true-in-content". "Truth-as-anticipation" and "true-as-anticipation" are abbreviated as "TAA".)

The strong notion is what allows Brouwer to claim that "truth is only in reality", i.e., TIC is correspondence to an actual construction. Not only is an object, a, a construction, so is a property-ascription, $P(a)$, and a reduction to absurdity, $\neg P(a)$ [Brouwer, 1907]. To take a simple example (not Brouwer's own), a natural number n is a construction consisting of n elements; the property-ascription *n is the sum of two primes* is a one-to-one and onto mapping of these n elements to the elements of two primes; and the reduction to absurdity of the same property-ascription would consist in attempted constructions of such mappings between the n elements and all pairs of two primes smaller than n that are executed as far as possible until "the construction no longer *goes*" (p. 127). These mappings are themselves constructions, described by Brouwer [1908] as the predicate being "embedded" into the object.

If, on the other hand, the subject employs (intuitionistic) logic to deduce new truths from existing truths, the new truths are not necessarily TIC. It is just that the subject now knows how to make them true. He has an algorithm which will produce the truth-maker for the sentence and he knows in advance of executing it, that it will have that result. In other words, he can anticipate the TIC of the sentence; it is TAA:

> [T]here is a system of general rules called *logic* enabling the subject to deduce from systems of word complexes conveying truths, other word complexes generally conveying truths as well. [...] This does not mean that the additional word complexes in question convey truths *before* these truths have been experienced [...] [Brouwer, 1948, 1243, emphasis original][9]

Let us consider a few examples. The proposition that 17 is an odd number, is TIC when a mapping between 17 and two copies of 8 and a unit has been constructed. The proposition that $10^{10} + 1$ is an odd number, however, is not TIC unless the subject has been extremely industrious, but it is TAA if the subject knows how to construct that number and a mapping between it and two copies of $5 \cdot 10^9$ and a unit. Further, assuming that our subject is not an expert on prime numbers, "$10^{10} + 1$ is prime" is neither TIC nor TAA. He does have an algorithm for deciding the proposition (let us assume that he knows that much), but he does not know the result of executing it in advance. That brings us to the case of complex sentences: "$10^{10} + 1$ is prime or $10^{10} + 1$ is composite" is TAA, for the subject has an algorithm which he knows will make the sentence TIC. He just does not know how; he does not know which disjunct will become TIC, so neither of the disjuncts are TAA.

Based on this discussion, we can give the first part of a more precise, recursive formulation of when a sentence is TIC and TAA, respectively:

- $P(a)$ is TIC iff P has been embedded into a;
- $\neg P(a)$ is TIC iff all options for embedding P into a have met an obstacle;

[9] See also [Brouwer, 1952, 141].

- $\phi \vee \psi$ is TIC iff ϕ is TIC or ψ is TIC (and the conclusion has been drawn);

◦ $P(a)$ is TAA iff an algorithm has been made which can make $P(a)$ TIC;

◦ $\phi \vee \psi$ is TAA iff an algorithm has been made which can make either ϕ TAA or ψ TAA.

Here "algorithm" means a method that not only is finite and would have the stated result if executed, but is known to the subject to be finite and to lead to that result. The notion of "embedding" is one that I will leave relatively vague as it is. Let me just make it clear that the idea is that TIC is an ontological rather than epistemological notion; when the subject has made a sentence TIC, he has not *verified* that it is true, he has *made* it true. (TAA on the other hand is connected with verifications.)

The clauses for conjunction are obvious, as are the generalizations of the clauses for the atomic sentences to predicates of arity more than 1:

- $\phi \wedge \psi$ is TIC iff ϕ is TIC and ψ is TIC;

◦ $\phi \wedge \psi$ is TAA iff an algorithm has been made which can make both ϕ and ψ TAA;

- $P(a_1, \ldots, a_n)$ is TIC iff P has been embedded into $\langle a_1, \ldots, a_n \rangle$;

- $\neg P(a_1, \ldots, a_n)$ is TIC iff all options for embedding P into $\langle a_1, \ldots, a_n \rangle$ have met an obstacle;

◦ $P(a_1, \ldots, a_n)$ is TAA iff an algorithm has been made which can make $P(a_1, \ldots, a_n)$ TIC.

For existential quantification the clauses are analogous to those for disjunction:

- $\exists x \phi(x)$ is TIC iff $\phi(a)$ is TIC for some object a in the domain;

◦ $\exists x \phi(x)$ is TAA iff an algorithm has been made which can construct an object a in the domain and make $\phi(a)$ TAA.

The special BHK-interpretation of the universal quantifier and conditionals fits perfectly into the present interpretation on the side of TAA:

◦ $\forall x \phi(x)$ is TAA iff an algorithm has been made which can turn any object a for which it is TAA that a is in the domain, into the TAA of $\phi(a)$;

◦ $\phi \to \psi$ is TAA iff an algorithm has been made which can turn TAA of ϕ into TAA of ψ.

Truth-in-content is an extensional notion, and therefore the clause for TIC of a universally quantified sentence must be as follows:

- $\forall x\phi(x)$ is TIC iff all the objects a in the domain have been constructed and $\phi(a)$ is TIC for them all.

This has the consequence that when the domain is infinite, a universally quantified sentence cannot be TIC, only TAA.

It is a more delicate matter what to say about TIC of conditionals. One thought would be that intuitionistic conditionals can only be understood in the algorithmic sense of the BHK-interpretation. In that case, TIC should never be attributed to conditionals. Insofar as it should, being an extensional notion, it seems most reasonable to understand it in terms of the classical definition of the conditional:

- $\phi \to \psi$ is TIC iff $\neg\phi$ is TIC or ψ is TIC.

Adopting this clause implies accepting the inference from $\neg\phi$ to $\phi \to \psi$ for any ϕ and ψ. There is no indication in Brouwer's writings that he does; see [van Atten, 2009] (where it also noted that Heyting, Kolmogorov, Troelstra, van Dalen, Martin-Löf and Dummett do accept it).

The TIC of $\neg\phi$ has only been defined for atomic ϕ so far, which must be remedied. But again, the extensionality of TIC, i.e. the requirement of correspondence with actual constructions, settles the issue unequivocally. A negation of a complex sentence being TIC must be understood in terms of the classical equivalence with a sentence where negations have narrower scope, so as to be reducible to the already defined TIC of atomic sentences and negations of atomic sentences:

- $\neg\neg\phi$ is TIC iff ϕ is TIC;
- $\neg(\phi \vee \psi)$ is TIC iff $\neg\phi$ is TIC and $\neg\psi$ is TIC;
- $\neg(\phi \wedge \psi)$ is TIC iff $\neg\phi$ is TIC or $\neg\psi$ is TIC;
- $\neg(\phi \to \psi)$ is TIC iff ϕ is TIC and $\neg\psi$ is TIC;
- $\neg(\exists x\phi x)$ is TIC iff all the objects a in the domain has been constructed and $\neg\phi a$ is TIC for them all;
- $\neg(\forall x\phi x)$ is TIC iff $\neg\phi a$ is TIC for some object a in the domain.

That just leaves us with TAA of negated sentences to be defined:

- ◦ $\neg\phi$ is TAA iff an algorithm has been made which can turn TAA of ϕ into some construction and the obstruction of the same construction.

This interpretation of Brouwer is, in a sense, a combination of the three proposed interpretations I discussed above: truth as actual construction, truth as potential construction and truth as proof. Both TIC and TAA consist of actual constructions. TIC consists in the actual construction of "things" and "qualities of things" in the language of the quotation at the beginning of this paper, and TAA consists in the actual construction of "rules". As such, both notions of truth are tensed. On the other hand, the admission of the weaker[10] notion of truth, TAA, is due to a reliance on potential constructions. It is a trust in the possibility of, to a certain extent, predicting the properties of not yet effected constructions which justifies anticipated-truth when there is not yet truth in the strong ontic sense. And finally, TAA is identified with the existence of proof. But it is in a sense of proof where it does not necessarily have to be a linguistic entity. Rather, an intuitionistic proof is a method for producing the truth-maker of the given sentence (although the subject needs to know that the method does that, and that knowledge may be the result of a proof in the traditional sense of the word). In this way, TAA is grounded in TIC. TAA of complex sentences is defined in terms of TAA of simpler sentences, and TAA of atomic sentences is defined in terms of TIC. Thus, there is no problem of impredicativity as in the BHK interpretation.

That a sentence is TAA means that it would become TIC if the algorithm in question were executed along with the algorithms corresponding to simpler sentences thereby produced and the... etc. down to atomic sentences.[11] However, when there are universal generalizations or negated existential claims over infinite domains involved, that task is impossible, as it consists in the execution of an infinity of algorithms. Still, the grounding of TAA in TIC is not thereby nullified, for any given finite part of the infinite conjunction, which such a sentence amounts to, can be realized as TIC.

The present interpretation accommodates both the [1951a] and the [1955] quotations above, without having to resort to the problematic "reduction" of truth to existence of proof. The four possibilities for the status of an assertion a in the former quotation are (1) that a is TIC, (2) that $\neg a$ is TIC, (3a) that neither a nor $\neg a$ is TIC but $a \vee \neg a$ is TAA, and (3b) that neither a nor $\neg a$ is TIC and $a \vee \neg a$ is not TAA. In the latter quotation, the four possibilities distinguished are (1) that a is TAA, (2) that $\neg a$ is TAA, (3) that neither a nor $\neg a$ is TAA but $a \vee \neg a$ is TAA, and (4) that neither a nor $\neg a$ nor $a \vee \neg a$ is TAA. These are different categorizations but they both give four possibilities which are mutually exclusive and collectively exhaustive.

With this interpretation we can also explain how Brouwer can deny in the π example that it was true in his time that a k_1 exists without denying the predetermination of the sequence of decimals of π. For any claim about the

[10]For any sentence ϕ, ϕ being TIC implies that ϕ is TAA. The TAA-making algorithm is the "empty" algorithm that is vacuously executed.

[11]In the case of universally quantified sentences, conditionals, and negations there are prerequisites for doing so: for universally quantified sentences one would need to have constructed the entire domain; for conditionals (assuming that the above clause for TIC of such is adopted) one would need the TAA of the antecedent; and for negations one would need, *per impossibile*, the TAA of the negated sentence.

value of a specific decimal, the subject can anticipate finding the answer, in the sense that he knows that he will find it within a preknown number of construction steps if he goes through the appropriate procedure. If he starts looking for a k_1 by searching through the decimals one by one, he cannot anticipate finding one, he can merely hope for it.

Even though the issue has already been touched upon, let me explicate the consequences for the semantics of disjunctions. The case of TIC is simple: if a disjunction is TIC then at least one of the disjuncts is too. But a disjunction can be TAA without any of the disjuncts being so. In particular $\tau \vee \neg\tau$ is TAA if the subject has a decision procedure for τ, but if he does not know in advance on which side the procedure will come out, neither τ nor $\neg\tau$ will be TAA. This allows Brouwer to state that

> Each assertion τ of the possibility of a construction of bounded finite character in a finite mathematical system furnishes a case of realization of the principle of the excluded third [Brouwer, 1948, 1245]

For in a finite system the procedure "try all possibilities" is a decision procedure; it will result in either τ or $\neg\tau$ becoming TIC and *a fortiori* TAA. So TAA does not distribute over disjunctions, but it is, so to speak, distribut*able* over disjunctions with a little work. That is, a disjunction being TAA at a given point in time does not imply that either of the disjuncts is TAA at that time, only that one of the disjuncts can be made TAA by executing the algorithm that makes the disjunction TAA.

We are not in possession of a decision procedure for Goldbach's Conjecture, $\forall n P(n)$, and hence $\forall n P(n) \vee \neg\forall n P(n)$ is not TAA. But for each n, we do have such a procedure for $P(n)$, making $P(n) \vee \neg P(n)$ TAA. So the algorithm consisting of "plugging" the given n into that procedure is the algorithm which turns any object in the domain $\mathbb{N}$ into the TAA of $P(n) \vee \neg P(n)$, required for $\forall n(P(n) \vee \neg P(n))$ being TAA.

It may also be worth explicating why the clause for TAA of a disjunction does not read "$\phi \vee \psi$ is TAA iff an algorithm has been made which can make either ϕ TIC or ψ TIC". Let again τ be a decidable but undecided proposition. Then this proposition, where n ranges over the natural numbers, is TAA (for anyone in possession of the decision procedure and aware of the following): $\forall n(\tau \wedge n = n) \vee \forall n(\neg\tau \wedge n = n)$. By deciding τ, one of the disjuncts becomes TAA but not TIC, for the latter would presuppose a completed construction of all the natural numbers.

3 An equivalence in propositional logic

In this section, and the following two sections, I will confront the interpretation with three examples of Brouwerian mathematics to show how it can account for them. The three examples are those that van Atten in his [2012] brings forward in defense of the claim that the "B" does rightly belong in the name "BHK-interpretation", and I have copied his subsection headings.

The first is Brouwer's proof of the logical law $\neg\neg\neg A \leftrightarrow \neg A$. This proof van Atten uses (sections 2.2 and 3.1.1) to argue against an understanding of conditionals $A \to B$ as meaning just "$A \wedge B$ with the extra information that the construction for B was obtained from that for A" (which would be a consequence of the truth-as-actual-construction interpretation[12]):

> The argument begins by pointing out that $A \to B$ implies that $\neg B \to \neg A$ [...] It would not have been possible for Brouwer to make this inference if at the time it would have been among his proof conditions of an implication to have a proof of the antecedent, as then a proof of $A \to B$ would lead to a proof of B and thereby make it impossible to begin establishing the second implication by proving its antecedent $\neg B$. [van Atten, 2012, section 3.1.1]

This is not an argument specifically for the BHK-interpretation, only against the mentioned alternative. Its conclusion is also consistent with the present interpretation, where $\neg\neg\neg A \leftrightarrow \neg A$ is TAA. Or rather: $\neg\neg\neg A \leftrightarrow \neg A$ is TAA for anyone who has understood the following proof (or one like it), as it provides a method of turning the TAA of $\neg\neg\neg A$ into the TAA of $\neg A$ and *vice versa*:

We first prove the TAA of $A \to \neg\neg A$. That is done by providing a method of turning TAA of A into TAA of $\neg\neg A$. So assume that A is TAA. TAA of $\neg\neg A$ is an algorithm for turning TAA of $\neg A$ into a construction and the obstruction of the same construction. So assume also that $\neg A$ is TAA. Use that to turn the TAA of A into a construction and the obstruction of the same construction. Discarding the second assumption, we have the TAA of $\neg\neg A$. And by also discarding the first assumption, the TAA of $A \to \neg\neg A$ is reached.

Second, we prove that the TAA of $A \to B$ implies the TAA of $\neg B \to \neg A$. Assume the antecedent and the TAA of $\neg B$. The following is an algorithm for turning the TAA of A into a construction and the obstruction of the same construction, i.e. the TAA of $\neg A$: use the first assumption to turn the TAA of A into the TAA of B and then use the second assumption to turn that into a construction and the obstruction of the same construction.

A special case of the first proposition proved is that $\neg A \to \neg\neg\neg A$ is TAA. And the two propositions together imply the TAA of $\neg\neg\neg A \to \neg A$. The algorithm which makes $\neg\neg\neg A \leftrightarrow \neg A$ TAA is then simply the concatenation of these two algorithms.

This example is one that the BHK-interpretation and the "two truths" interpretation can account for equally well. I will argue that the next two are some where the latter does better than the former.

[12]And a possible interpretation of the TIC of $A \to B$, instead of the one above.

In this section I have considered a single logical theorem. But, the more general claim that this interpretation is complete with respect to intuitionistic first-order predicate logic is also true. It is true in the following sense: for each axiom and inference rule of that logic there is a certain (quite simple) proof, such that the axiom/inference rule is TAA for anyone who has understood the proof.

4 The proof of the Bar Theorem

The next example considered by van Atten is the proof of the Bar Theorem [Brouwer, 1924b, 1927, 1954, 1981], which can be stated thus: *if B is a decidable bar on a spread, then B contains a well-ordered thin bar.* I will begin this section by explaining the terms used in this formulation, before interpreting Brouwer's proof of, and his comments on, it in the light of the two truths. As this will get somewhat abstract and perhaps difficult to follow, the section will conclude with a toy example to make matters more concrete.

For present purposes we can define a spread as a species of tuples of natural numbers (among which we count the empty tuple) which satisfies the following. First, it is decidable for any tuple whether or not it is in the spread. Second, if $\langle a_1, \ldots, a_n \rangle$ is in the spread then so is $\langle a_1, \ldots, a_{n-1} \rangle$. Third, if $\langle a_1, \ldots, a_n \rangle$ is in the spread then there exists (in the intuitionistic sense of the word) a natural number a_{n+1} such that $\langle a_1, \ldots, a_n, a_{n+1} \rangle$ is also in the spread.

A tuple $\langle a_1, \ldots, a_n \rangle$ is called an ascendant of a tuple $\langle a_1, \ldots, a_n, \ldots, a_{n+m} \rangle$ in the spread, and the latter is called a descendant of the former. If $m = 1$ the modifier "immediate" may be added.

It is helpful mentally to picture a spread as a tree in which each tuple is a node with all its immediate descendants as nodes immediately below it. An infinite route from the root, i.e. the empty tuple, downwards then corresponds to a choice sequence. In that case I will say that the choice sequence is in the spread.

A bar is a subspecies of a spread such that every choice sequence in the spread has an initial segment (one of the nodes it goes through) in that subspecies. A bar can be pictured as an area stretching the entire breadth of the tree so that every choice sequence must pass through it. It was discovered by Kleene and Vesley [1965] that for the theorem to hold it must be assumed that the bar is decidable, which is to say that it is decidable whether a given tuple is in it or not.

For the purpose of being a bar, such an area does not need to be deeper than one node; hence the notion of a thin bar which is a bar such that for any tuple in it, no ascendant or descendant of it is also in the bar.

Well-orderings are defined inductively. A one-element species is a well-ordering, and if $A_0, \ldots, A_n$ or $A_0, \ldots$ are disjoint well-orderings, then their union, equipped with the following ordering, is also a well-ordering: $x < y$ if either

x is from an A_i and y is from an A_j such that $i < j$ or x and y are from the same A_i and ordered $x < y$ therein. The A_is are called "constructional subspecies" of the resulting well-orderings.

For this analysis, there is one specific detail about the ontology of well-orderings that is important. Brouwer [1981, 44] demands that the A_is are in the "available stock" of already constructed well-orderings, before they can be used to construct a larger well-ordering. This may suggest a demand for strict "bottom-up" construction, which is misleading. First of all, one should of course remember that if a well-ordering is constructed out of infinitely many other well-orderings, then the infinity is potential, so they cannot all have been previously constructed. It follows that the existence of these well-orderings must be understood as true-as-anticipation.

In other words, Brouwer accepts that the larger-scale structure is constructed prior to the smaller-scale details of that structure. This conclusion is reinforced by an example from [Brouwer, 1981, 49], where a well-ordering is constructed with the aid of a fleeing property.[13] Pretend that such a property is given and let k be the (hypothetical) least natural number with that property. Further, let A_i be an ω-sequence for $i < k$ but just a one-element species for $i \geq k$. Brouwer takes these A_is to be acceptable building blocks for a well-ordering. So, not only can the building blocks be constructed after the house, we can also be largely ignorant about the shape and size of these building blocks.

This brings us to the end of the explanation of what the theorem says. We can turn to the interpretation of its proof.

The Bar Theorem has the form of an implication, and the proof turns on considerations of how possible proofs of the antecedent can be manipulated into a proof of the consequent. This fits with the BHK-interpretation. But, what Brouwer means by "proof" in that context is very different from the normal understanding of the word.[14] When his non-standard use of the word is taken into account, the two truths interpretation can explain Brouwer's proof in a way that is more detailed and less prone to misunderstandings than the BHK-interpretation.

Brouwer writes that any proof of the antecedent can be expanded into a so-called canonical proof. Such a canonical proof is an infinite (if the part of the tree "above" the bar is) mental construction with a structure that is itself a well-ordering. When the word "proof" is taken in the normal sense, that is a baffling claim.[15] How can something persuade us of the truth of a proposition if it is infinite and therefore unsurveyable? And how

[13] "A fleeing property is a property which can be proved for any given natural number either to hold or to be contradictory, whilst neither a number with that property is known, nor a proof that it is contradictory for every natural number" [Brouwer, 1933]. An example is (or was; see footnote 1) the property *a 0123456789-sequence begins at the nth decimal of* π.

[14] See [Sundholm and van Atten, 2008] regarding Brouwer's use of the words "proof", "demonstration", and "argument" and their equivalents in German and Dutch.

[15] For example, relying on a truth=proof interpretation of intuitionism, Epple [2000] comes to the conclusion that the proof of the Bar Theorem does not live up to Brouwer's own epistemological standards.

can Brouwer be certain that a proof must be expandable into the form he describes? The word "proof" (or "demonstration") must be understood differently:

> Intuitionistically, to give a demonstration of a mathematical theorem is not to produce a certain linguistic object, but to produce a mental mathematical construction (or a method to obtain one, which method is of course also a mental mathematical construction) that makes the corresponding proposition true. Therefore, the requirement, for a demonstration that the consequence
>
> $$\text{A is true} \Rightarrow \text{B is true}$$
>
> holds, of a method that transforms any demonstration that A is true into one that B is true, is really the requirement of a method that transforms any mathematical construction that makes A true into one that makes B true. [Sundholm and van Atten, 2008, 61]

This I believe to be correct, but also easy to misunderstand: is a "mathematical construction that makes A true" not an infinite structure which can never be completed and thereby ready for being transformed into a mathematical construction that makes B true? With the present interpretation, with its distinction between two ways that A and B can be true, we can make this more precise and avoid the confusion. A proof of the implication is a method for transforming the TAA of A into the TAA of B, not a method to transform the actually infinite structure that would make A TIC into the actually infinite structure that would make B TIC.

Getting to the specific details of the proof of this theorem, the antecedent says, when interpreted in the appropriate intuitionistic way, that when I construct a choice sequence in the spread, I will at some point construct an initial segment that is in the bar; that I will be able to determine that the initial segment is in the bar, when it is; and that I know in advance that it will happen within a calculable number of construction steps.[16] The antecedent is of the ∀∃-form; for each choice sequences in the spread, there exists an initial segment of it that is in the bar. So the antecedent being TAA means that for any given choice sequence, an initial segment of it that is in the bar can be constructed, i.e., that the instance of the universal generalization for the specific choice sequence can be made TIC. It is hopeless to expect that we could know what any possible proof, in the normal sense of that word, that could make that antecedent TAA would look like, but it is trivial to see how this TIC is accomplished: just construct the terms of the choice sequence one after another, and each time run the decision

[16] The last bit is not precise: "the algorithm in question may indicate the calculation of a maximal order n_1 at which will appear a finite method of calculation of a further maximal order n_2 at which will appear a finite method of calculation of a further maximal order n_3 at which will appear a finite method of calculation of a further maximal order n_4 at which the postulated node of intersection must have been passed. And much higher degrees of complication are thinkable." [Brouwer, 1954, 13]

procedure for bar-membership on the resulting initial segment. The infinite mental construction, that Brouwer misleadingly calls a canonical proof, is the (unaccomplishable) TIC of all the instances, i.e. of the antecedent.

Therefore, what Brouwer means by "canonical proof" is very different from how Dummett understands the term. The reasoning that makes the subject know in advance of constructing a choice sequence, that it will meet the bar, is not the thing that can be transformed into a canonical form. Rather, the canonical form is the imagined infinite result of applying an ability to construct a certain kind of objects (choice sequences hitting bars) in all possible ways, and that ability can be transformed into an ability to construct another kind of objects (well-orderings).

Let me elaborate in a way that brings us closer to Brouwer's own formulations. He calls an element of the bar a "secured" node, while a node above is "unsecured" but is "securable" when it is established that all choice sequences through it hit a secured node. The securability of a node is accomplished through induction from below; if all the immediate descendants of a given node are securable, then that node is securable. That is the induction step and is called an "elementary inference". The securability of nodes high up in the tree, in particular the root, is reached through repeating such steps; if the tree is large enough, infinitely many of them. In his formulations Brouwer indulges in the fantasy that we could actually go through the entire construction process, bottom-up, but of course he should not be interpreted literally; it is a mere *façon de parler*. Actually, we can only construct finitely many nodes, top-down, and must have some means, independently of actually constructing *all* the nodes that contribute to their securability, of knowing that all choice sequences through them will hit the bar. Such means are of course arguments or, in the normal meaning of the word, proofs.

The strong similarities between constructing well-orderings and establishing securability of nodes should now be clear. First of all, the elements of the thin bar correspond to the one-element well-orderings, and the nodes above it to the well-orderings that are constructed from smaller well-orderings. In addition, for both securability and well-orderings, the literal reading of Brouwer suggests a strict bottom-up constructional process in which a brick in the wall can only be added, when all the bricks it rests on are in place; but instead it must be seen as a potentially infinite top-down process where the status of being securable/being a well-ordering stems from prior knowledge that whatever choice sequence/series of constructional subspecies of constructional subspecies of constructional subspecies etc. the subject may *actually* construct, it will hit the bar/bottom out in a one-element well-ordering.

The ability that the subject must possess (and know himself to possess) in order for the antecedent to be TAA is virtually the same as the ability that makes the consequent TAA. Hence, proving the Bar Theorem is actually trivial. To prove the TAA of the implication, what is needed is a way of turning any method for constructing any given "part" of the truth-maker ("TIC-maker") of the antecedent into a method for constructing any given

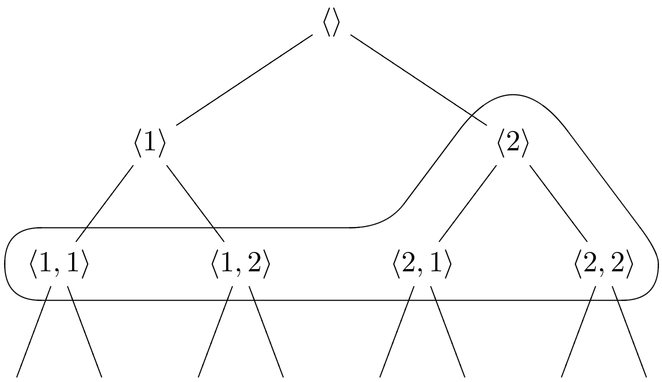

Figure 1

part of the truth-maker of the consequent. But doing the former is essentially the same as doing the latter, so any method for the former is almost a method for the latter, making the "turning" of the one into the other trivial.

Brouwer himself notes this triviality in [1927, original text: 63fn7, English translation: 460fn7]. That is a comment that can be explained with the two-truths interpretation, and this explanatory success is, I believe, a point in its favor.[17]

Now to the promised toy example. In Figure 1 is the top part of a spread where the natural numbers allowed are restricted to 1 and 2. It is equipped with a very simple five element bar which contains a well-ordered three element thin bar. According to the bottom-up story, the subject should construct the well-ordered thin bar by first constructing $\langle 1,1\rangle$, $\langle 1,2\rangle$ and $\langle 2\rangle$, then constructing $(\langle 1,1\rangle,\langle 1,2\rangle)$ and finishing with $((\langle 1,1\rangle,\langle 1,2\rangle),\langle 2\rangle)$ (the brackets indicate constructional subspecies). Of course that is quite feasible in this case, the thin bar being finite. But for the purpose of the toy example, let us pretend that the subject (for whatever mundane reason) only has time to do two construction steps, and let that restriction simulate finitude. Then the existence of the bar and the existence of the well-ordered thin bar cannot become TIC.

But he can construct any part of the bar, top-down. For instance the first construction step could result in $\{\langle 1\rangle^\star,\ldots\}$ and the second in $\{\langle 1,2\rangle,\ldots\}$ (the curly brackets indicate a species (as a bar is) with some intensional criterion of membership not displayed; the dots indicate that it is incomplete qua extensional object; and the star indicates that $\langle 1\rangle$ is not itself an element of the bar but has to be further developed). The ability to "fill out" any part of this species and the knowledge that any "starred element" can be developed into an element of the species no matter which natural number (here 1 or 2) is chosen in the following steps is what constitutes the TAA of the antecedent of the Bar Theorem.

[17] The above discussion could perhaps have been improved in precision with a formalization of the Bar Theorem. But it is not clear to me that such a formalization is possible. An attempt at a formalization can be found in [Kleene and Vesley, 1965, 52], but there the consequent is rendered as a principle of backward induction, which I think is unfaithful to Brouwer.

That ability and knowledge is, as noted, virtually the same as the the ability and knowledge that constitutes the TAA of the consequent. Here the corresponding top-down construction of a part of the well-ordering has as its first step $(\langle 1\rangle^{\star}, \ldots)$ and as its second $((\ldots, \langle 1, 2\rangle), \ldots)$. The difference is simply that in the construction of (parts of) the well-ordered thin bar, some extra structure from the construction process is preserved.

5 Ordering axioms

The last example [van Atten, 2012, section 3.1.3] is concerned with Brouwer's definition of so-called "virtual orderings". These are given through five axioms, of which one, serving as example, will be sufficient for present purposes; so let us take the simplest, number five: "From $r < s$ and $s < t$ follows $r < t$". The following comment of Brouwer's, concerning these axioms, is cited by van Atten as a confirmation of the BHK-interpretation:

> The axioms II through V are to be understood in the constructive sense: if the premises of the axiom are satisfied, the virtually ordered set should provide a construction for the order condition in the conclusion.

Van Atten claims that "[t]his is a clear instance of the clause for implication in the Proof Interpretation". But the BHK-interpretation renders the axiom as "any proof of $r < s$ and $s < t$ must be convertible into a proof of $r < t$". However, the operative words in Brouwer's comment are "satisfied" and "provide a construction" which are more specific than the ambiguous "proof". The "two truths" interpretation captures this comment much better. For the TAA of the axiom, i.e.

TAA of (from $r < s$ and $s < t$ follows $r < t$)

is equivalent to

TAA of ($r < s$ and $s < t$) can be turned into TAA of $r < t$,

which is the same as

TAA of ($r < s$ and $s < t$) can be turned into an algorithm which makes $r < t$ TIC.

This seems to be a much more reasonable explicitation of the comment.

6 Two-truths versus BHK

In the original sixth-century Indian version of chess, the winning criterion was actually to capture the opponent's king. Only later did the Persians amend the rules so that a player would win already when the king was made check mate. The original version is the most intuitive and it would be difficult to imagine that the game could have been invented directly in the

Persian form; the concept of check mate is difficult to explain except when done in terms of what *would happen* in the next round of the game. On the other hand, the Persian "contraction" of the game makes good sense, as that final round is trivial and not worth actually executing.

This makes for a nice analogy: TIC is like actually capturing the king, while TAA corresponds to making the king check mate or, more generally, being in possession of a winning strategy for the Indian version of the game. If the existence of such a winning strategy is common knowledge to the players, then there is no point in actually completing the game – but only by reference to the possibility of actually capturing the king does the winning strategy make sense.

It is a problem with Brouwer's writings (and those of his interpreters) that this distinction is not made clearly. In chess, the "contraction" of the game is so simple that anyone presented with the Persian version can easily see the connection with the Indian version, and therefore think like an Indian while acting like a Persian. The "contraction" of TIC to TAA is quite complex. Therefore, an introduction to intuitionism should begin by clearly explaining TIC and only then move on to the less basic and more abstract concept of TAA, which is what the proofs in intuitionistic papers make contact with.

The clauses of the BHK-interpretation conflate the two kinds of truth: the clause for disjunction is only correct for TIC; while the clauses for the conditional and the universal quantifier are only correct for TAA. I think the BHK-interpretation is comparable to early analysis in that only experts can interpret the interpretation the right way. Anyone learning about it for the first time is almost bound to get it wrong.

There is an obvious problem for my interpretation and the critique of BHK that needs to be considered, namely that Brouwer explicitly endorsed Heyting's interpretation:

> [W]hile preparing a note on intuitionism for the Bulletin of the Royal Academy of Belgium, I was pleasantly surprised to see the publication of a note of my student Mr. Heyting which elucidates in a magisterial manner the points that I wanted to shed light upon myself. I believe that after Heyting's note little remains to be said. [van Dalen, 2013, 607]

There are a couple of reasons why I do not attribute much weight to this endorsement. First, given that formalism is not something Brouwer cares for, he could easily have made that remark without having given it full consideration. And even if he did, it is a commonplace that you find out that a given formulation of your position, that you first thought to be perfectly precise, turns out to be improvable. Also, Brouwer's stamp of approval on Heyting's clauses was within the context of a discussion about whether intuitionism introduces a third truth value.[18] Thus, it is not unreasonable to assume that the approval is primarily due to the fact that Heyting's interpretation was accurate in that particular respect.

[18]Thanks to Göran Sundholm for pointing this out.

7 Lawless choice sequences

By qualifying the notion of "algorithm", we can extend the two-truths interpretation to also cover choice sequences that are not lawlike. Take the example of a choice sequence α, of which only the first three elements have been chosen and the sole restriction on future choices is that they must be natural numbers. Consider the following example sentences:

1. The 17^{th} term of α is 99.
2. The 17^{th} term of α is not 99.
3. The 17^{th} term of α is 99 or the 17^{th} term of α is not 99.

Neither of these sentences is TIC. Brouwer would say that the last sentence is true [1908], while the first two sentences are not. With the right understanding of "algorithm", this fits with the given clauses for TAA.

With an understanding of the word that is too rigid, sentence 3 would come out as not TAA. The procedure that makes the sentence TIC is the one described by the instruction "choose additional 14 elements of α". On a narrow understanding this is not an algorithm because it involves choices.

On the other hand, we cannot replace "algorithm" with something as broad as "method", for that would over-generate sentences that are TAA: the subject has a method for making sentence 1 TIC, namely deciding to pick 99 as the 17^{th} element.

The right understanding is most clearly explained with a story about *two* subjects. One subject is the generator of α and chooses one new element thereof, whenever he is prompted to do so by the second subject. The second subject is the one for which sentences 1 and 2 are not TAA while sentence 3 is. She has an algorithm, in the strict sense of the word, that will make sentence 3 TIC (by making one of the disjuncts TIC), namely simply *14 times in a row, ask the first subject for a new element of α.* Hence, sentence 3 is TAA for her, while sentences 1 and 2 are not, as she has no influence on what numbers are chosen.

Having to refer to two different subjects is not in the spirit of Brouwer who emphasizes the individual. It can be avoided if we imagine a subject who manages to keep his tasks separate – that is, when he chooses elements of a choice sequence, he chooses freely within the explicit restrictions he has imposed on himself without being influenced by the judgments he himself has previously made about that same choice sequence.

Acknowledgments

This paper has been greatly improved because of help from Mark van Atten, Crispin Wright, Øystein Linnebo, Dag Prawitz, audiences in Odense and Aberdeen, and the anonymous referees.

References

P. Benacerraf and H. Putnam, editors. *Philosophy of Mathematics: Selected Readings*. Prentice-Hall, 1964.

L. E. J. Brouwer. *Over de Grondslagen der Wiskunde*. PhD thesis, Amsterdam, 1907. Translated as "On the Foundations of Mathematics" in Brouwer [1975], 11–101.

L. E. J. Brouwer. De onbetrouwbaarheid der logische principes. *Tijdschrift voor wijsbegeerte*, 2:152–158, 1908. Translated as "The Unreliability of the Logical Principles" in Brouwer [1975], 107–111.

L. E. J. Brouwer. Intuitionism and formalism. *Bulletin of the American Mathematical Society*, 20:81–96, 1913. Reprinted in Brouwer [1975], 123–138.

L. E. J. Brouwer. Intuitionistische Zerlegung mathematischer Grundbegriffe. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 33:251–256, 1924a. Reprinted in Brouwer [1975], 275–280.

L. E. J. Brouwer. Beweis, dass jede volle Funktion gleichmässig stetig ist. *Koninklijke Akademie van wetenschappen te Amsterdam, Proceedings of the section of sciences*, 27:189–193, 1924b. Reprinted in Brouwer [1975], 286–291.

L. E. J. Brouwer. Über Definitionsbereiche von Funktionen. *Mathematische Annalen*, 97:60–75, 1927. Reprinted in Brouwer [1975], 390–405. Translated as "On the Domains of Definition of Functions" in van Heijenoort [1967], 446–463.

L. E. J. Brouwer. Intuitionistische Betrachtungen über den Formalismus. *Sitzungsberichte der Preussischen Akademie von Wissenschaften*, pages 48–52, 1928. Reprinted in Brouwer [1975], 409–414.

L. E. J. Brouwer. Willen, weten, spreken. *Euclides*, 9:177–193, 1933. Translated as "Will, knowledge and speech" in W. P. van Stigt (1990), Brouwer's Intuitionism, 418–431. North-Holland.

L. E. J. Brouwer. Richtlijnen der intuitionistische wiskunde. *Koninklijke Akademie van wetenschappen te Amsterdam, Proceedings of the section of sciences*, 50:339, 1947. Translated as "Guidelines of intuitionistic mathematics" in Brouwer [1975], 477.

L. E. J. Brouwer. Consciousness, philosophy, and mathematics. In *Proceedings of the 10th International Congress of Philosophy*, pages 1235–1249, 1948. Reprinted in Brouwer [1975], 480–494.

L. E. J. Brouwer. Notes for a lecture. In *van Stigt [1990]*, 447–452. 1951a.

L. E. J. Brouwer. Changes in the relation between classical logic and mathematics. In *van Stigt [1990]*, 453–458. 1951b.

L. E. J. Brouwer. Historical background, principles and methods of intuitionism. *South African Journal of Science*, 49:139–146, 1952. Reprinted in Brouwer [1975], 508–515.

L. E. J. Brouwer. Points and spaces. *Canadian Journal of Mathematics*, 6: 1–17, 1954. Reprinted in Brouwer [1975], 522–538.

L. E. J. Brouwer. The effect of intuitionism on classical algebra of logic. *Proceedings of the Royal Irish Academy*, 57:113–116, 1955. Reprinted in Brouwer [1975], 551–554.

L. E. J. Brouwer. *Collected Works 1*. North-Holland, 1975. Edited by A. Heyting.

L. E. J. Brouwer. *Brouwer's Cambridge Lectures on Intuitionism*. Cambridge University Press, 1981. Edited by D. van Dalen.

M. Dummett. *Truth and Other Enigmas*. Duckworth, 1978a.

M. Dummett. The philosophical basis of intuitionistic logic. In *Dummett [1978a]*, 215–247. 1978b.

M. Dummett. *Elements of Intuitionism*. Oxford University Press, second edition, 2000.

M. Epple. Did Brouwer's intuitionistic analysis satisfy its own epistemological standards? In V. F. Hendricks, S. A. Pedersen, and K. F. Jørgensen, editors, *Proof Theory: Historical and Philosophical Significance*. Kluwer Academic Publishers, 2000.

A. Heyting. Sur la logique intuitionniste. *Académie Royale de Belgique, Bulletin*, 16:957–963, 1930a. Translated as "On Intuitionistic Logic" in Mancosu [1998], 306–310.

A. Heyting. Die formalen Regeln der intuitionistischen Logik. *Sitzungsberichte der Preussischen Akademie von Wissenschaften, Physikalisch-mathematische Klasse*, pages 42–56, 1930b. Translated as "The Formal Rules of Intuitionistic Logic" in Mancosu [1998], 311–327.

A. Heyting. Die intuitionistische Grundlegung der Mathematik. *Erkenntnis*, 2:106–115, 1931. Translated as "The Intuitionist Foundations of Mathematics" in Benacerraf and Putnam [1964].

A. Heyting. *Mathematische Grundlagenforschung, Intuitionismus, Beweistheorie*. Verlag von Julius Springer, 1935.

S. C. Kleene and R. E. Vesley. *The Foundations of Intuitionistic Mathematics*. North-Holland, 1965.

A. N. Kolmogorov. Zur Deutung der intuitionistischen Logic. *Mathematische Zeitschrift*, 35:58–65, 1932. Translated as "On the Interpretation of Intuitionistic Logic" in Mancosu [1998], 328–334.

S. Kripke. *Wittgenstein on Rules and Private Language.* Basil Blackwell, 1982.

P. Mancosu. *From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s.* Oxford University Press, 1998.

K. Mulligan, P. Simons, and B. Smith. Truth-makers. *Philosophy and Phenomenological Research*, 44:287–321, 1984.

D. Prawitz. On the idea of a general proof theory. *Synthese*, 27:63–77, 1974.

P. Raatikainen. Conceptions of truth in intuitionism. *History and Philosophy of Logic*, 25:131–145, 2004.

G. Sundholm and M. van Atten. The proper explanation of intuitionistic logic: On Brouwer's demonstration of the Bar Theorem. In M. van Atten, P. Boldini, M. Bourdeau, and G. Heinzmann, editors, *One Hundred Years of Intuitionism (1907-2007).* Birkhäuser, 2008.

A. S. Troelstra. Analysing choice sequences. *Journal of Philosophical Logic*, 12:197–260, 1983.

M. van Atten. *On Brouwer.* Wadsworth, 2004.

M. van Atten. On the hypothetical judgement in the history of intuitionistic logic. In C. Glymour, W. Wang, and D. Westerstahl, editors, *Logic, Methodology, and Philosophy of Science XIII: Proceedings of the 2007 International Congress in Beijing*, pages 122–136, 2009.

M. van Atten. The development of intuitionistic logic. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* Fall 2012 edition, 2012.

D. van Dalen. *L.E.J. Brouwer en de Grondslagen van de Wiskunde.* Epsilon, 2001.

D. van Dalen. *L.E.J. Brouwer: Topologist, Intuitionist, Philosopher.* Springer, 2013.

J. van Heijenoort. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931.* Harvard University Press, 1967.

W. P. van Stigt. *Brouwer's Intuitionism.* North-Holland, 1990.

D. Wells. *The Penguin Dictionary of Curious and Interesting Numbers.* Penguin Books, 1986.